ELECTROMAGNETIC PRELIMINAIRES

Non relativistic ($v^2/c^2 << 1$) Lorentz transformation of $\boldsymbol{E}$ and $\boldsymbol{B}$.

$$\boldsymbol{E}' = \boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}, \boldsymbol{B}' = \boldsymbol{B} - \frac{\boldsymbol{v}\times\boldsymbol{E}}{c}.$$

$\boldsymbol{E}', \boldsymbol{B}'$ are the fields experienced in the reference frame with velocity $\boldsymbol{v}$ relative to the frame in which the fields are $\boldsymbol{E},\boldsymbol{B}$, respectively.

A moving physical system experiences only the fields $\boldsymbol{E}', \boldsymbol{B}'$ in its own reference frame.

There is a magnetic field of about half a Gauss filling this room.

Is there an electric field in this room?

Magneto Hydrodynamics

MHD is based on the concept that the magnetic field is transported by the fluid

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \text{dissipation terms.}$$

Consider a gas with enough free electrons and ions that it cannot support any significant electric field **E**' in its own frame of reference.

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}$$

If **E**'=0, then

$$\mathbf{E} = -\frac{\mathbf{v}}{c} \times \mathbf{B}$$

and

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

The result is MHD, regardless of the details

The Poynting vector

$$\boldsymbol{P} = c\frac{\boldsymbol{E}\times\boldsymbol{B}}{4\pi}$$

becomes

$$\boldsymbol{P} = \boldsymbol{v}_{\perp}\frac{\boldsymbol{B}\cdot\boldsymbol{B}}{4\pi}$$

because

$$\boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B}/c$$

The Poynting vector asserts that the magnetic enthalpy is transported with the perpendicular plasma velocity.

The magnetic field is transported bodily with the plasma because there is no electric field in the frame of reference of the moving plasma.

Note that **E** plays no significant dynamical role.

$$\frac{E^2}{8\pi} = O\left(\frac{v^2}{e^2}\right)\frac{B^2}{8\pi}$$

Note, too, that the existence of **E** depends upon what frame of reference the calculation uses.

See example in V. Vasyliunas, 2001, Geophys. Res. Letters, **28**, 2177.

Similarly **j** plays no dynamical role because it has no energy and no strength.

Note that in any real gaseous medium **j** is driven by a weak **E**', pulling energy out of the magnetic field.

B causes **j**, not vice versa.

Note that **j** is driven by the relation

$$\frac{\partial \mathbf{E}}{\partial t} = c\nabla \times \mathbf{B} - 4\pi\mathbf{j}$$

Note that the dynamics is all in terms of

$$\rho = \text{NM}, \quad \mathbf{p} = \text{NkT}, \quad \mathbf{v}, \quad \text{and}\,\mathbf{B}$$

when $\mathbf{E} = -\mathbf{v}\times\mathbf{B}/c$

E and j are secondary passive quantities.

Consider the role of the neglected $\mathbf{E}'$.

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) - c\nabla\times\mathbf{E}'$$

For a collision dominated plasma, $\mathbf{j} = \sigma\,\mathbf{E}'$.

Hence $\mathbf{E}' = \frac{c}{4\pi\sigma}\nabla\times\mathbf{B}$ and

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) - \nabla\times(\eta\nabla\times\mathbf{B})$$

$$\eta = \frac{c^2}{4\pi\sigma} \sim 0.5\times10^{13}/\text{T}^{\frac{3}{2}}\,\text{cm}^3/\text{sec}$$

Magnetic Reynolds number

$$\text{N}_R = \frac{\text{vL}}{\eta}$$

For large N_R the principal effect is bulk transport of **B**.

For a partially ionized gas

N = number density of neutral atoms,
n = number density of ions/electrons

v = mean bulk velocity of neutral gas

w = mean bulk velocity of ions

u = mean bulk velocity of electrons

τ_i = ion-neutral collision time

τ_e = electron-neutral collision time

τ = ion-electron collision time

p = pressure of neutral gas

$$NM\frac{d\mathbf{v}}{dt} = -\nabla p + \frac{nM(\mathbf{w}-\mathbf{v})}{\tau_i} + \frac{nm(\mathbf{u}-\mathbf{v})}{\tau_c}$$

Consider a slightly ionized gas, $n<<N$.

Neglect ion and electron pressures.

$$m\frac{d\mathbf{u}}{dt} = -e\left(\mathbf{E} + \frac{\mathbf{u}\times\mathbf{B}}{\mathbf{c}}\right) - \frac{\mathrm{m}(\mathbf{u}-\mathbf{v})}{\tau_c} - \frac{\mathrm{m}(\mathbf{u}-\mathbf{w})}{\tau}$$

$$\mathrm{M}\frac{d\mathbf{w}}{dt} = +e\left(\mathbf{E} + \frac{\mathbf{w}\times\mathbf{B}}{\mathbf{c}}\right) - \frac{\mathrm{m}(\mathbf{w}-\mathbf{v})}{\tau_i} + \frac{\mathrm{m}(\mathbf{u}-\mathbf{w})}{\tau}$$

$$\mathbf{j}=\mathrm{ne}(\mathbf{w}-\mathbf{u})$$

From Ampere's law

$$\mathbf{u} = \mathbf{w} - \frac{\mathbf{c}}{4\pi ne}\nabla\times\mathbf{B}$$

Neglect the electron and ion inertia. The sum of the two eqns. of motion gives

$$\frac{\mathrm{nM}(\mathbf{w}-\mathbf{v})}{\tau_i} + \frac{\mathrm{nm}(\mathbf{u}-\mathbf{v})}{\tau_e} = \mathrm{en}(\mathbf{w}-\mathbf{u})\times\frac{\mathbf{B}}{c}$$

$$= \mathbf{j}\times\mathbf{B}/c = \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi}$$

Hence, for the neutral atoms

$$\mathrm{NM}\frac{\mathrm{d}\,\mathbf{v}}{\mathrm{dt}} = -\nabla p + \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi}$$

which is the usual MHD momentum eqn.

Note that

$$\mathbf{w} = \mathbf{v} + \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi n Q} + \frac{c\,m/\tau_e}{4\pi neQ}\nabla\times\mathbf{B}$$

$$\mathbf{u} = \mathbf{v} + \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi n Q} - \frac{c\,M/\tau_i}{4\pi neQ}\nabla\times\mathbf{B}$$

where

$$Q \equiv \frac{M}{\tau_i} + \frac{m}{\tau_e} \cong \frac{M}{\tau_i}$$

Then

$$\mathbf{E} = -\frac{\mathbf{v}\times\mathbf{B}}{\mathbf{c}} - \frac{[(\nabla\times\mathbf{B})\times\mathbf{B}]\times\mathbf{B}}{4\pi ncQ} + \frac{M/\tau_i - m/\tau_e}{4\pi neQ}(\nabla\times\mathbf{B})\times\mathbf{B}$$

$$+\frac{c}{4\pi ne^2}\left[\frac{(M/\tau_i)(m/\tau_e)}{Q} + \frac{m}{\tau}\right]\nabla\times\mathbf{B}$$

Define

$$\alpha \equiv \frac{cB}{4\pi ne}\left[\frac{M/\tau_i - m/\tau_e}{M/\tau_i + m/\tau_e}\right]$$ Hall coefficient

$$\beta \equiv \frac{B^2}{4\pi nQ}$$ Pedersen coefficient

$$\eta \equiv \frac{c^2}{4\pi ne^2}\left[\frac{(M/\tau_i)(m/\tau_e)}{M/\tau_i + m/\tau_e} + \frac{m}{\tau}\right]$$ Ohm's coefficient

Write **b** = **B**/B, so that

$$\mathbf{E} = \frac{\mathrm{B}}{\mathrm{c}}\left[-\mathbf{v}\times\mathbf{b} - \beta[(\nabla\times\mathbf{b})\times\mathbf{b}]\times\mathbf{b} + \alpha(\nabla\times\mathbf{b})\times\mathbf{b} + \eta\nabla\times\mathbf{b}\right]$$

$$\mathbf{E}' = \frac{\mathrm{B}}{\mathrm{c}}\left[\eta\nabla\times\mathbf{b} + \alpha(\nabla\times\mathbf{b})\times\mathbf{b} - \beta[(\nabla\times\mathbf{b})\times\mathbf{b}]\times\mathbf{b}\right]$$

The induction equation

$$\partial\mathbf{B}/\partial t = -c\nabla\times\mathbf{E}$$

becomes

$$\frac{\partial\mathbf{b}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{b}) - \nabla\times(\eta\nabla\times\mathbf{b}) + \nabla\times\{\beta[(\nabla\times\mathbf{b})\times\mathbf{b}]\times\mathbf{b} - \alpha(\nabla\times\mathbf{b})\times\mathbf{b}\}$$

This is the usual MHD eqn. with two extra terms.

In terms of the non-dimensional Lorentz force

$$\mathbf{f} = \frac{(\nabla\times\mathbf{b})\times\mathbf{b}}{4\pi}$$

$$\frac{\partial\mathbf{b}}{\partial \mathrm{t}} = \nabla\times(\mathbf{v}\times\mathbf{b}) - \nabla\times(\eta\nabla\times\mathbf{b}) + \nabla\times[\beta\mathbf{f}\times\mathbf{b} - \alpha\mathbf{f}]$$

The magnetic energy equation can be written

$$\frac{\partial}{\partial t}\left(\frac{1}{8\pi}\mathbf{b}^2\right)+\nabla\cdot\left[\frac{\mathbf{v}_\perp\mathbf{b}^2}{4\pi}+\eta\mathbf{f}+\beta\mathbf{b}^2\mathbf{f}+\alpha\mathbf{f}\times\mathbf{b}\right]$$

$$=-\mathbf{v}\cdot\mathbf{f}-\frac{\eta(\nabla\times\mathbf{b})^2}{4\pi}-4\pi\beta\mathbf{f}^2$$

The right hand side represents the dissipation of magnetic energy. The term in square brackets represents the transport of magnetic energy.

Consider the Hall effect with $\beta=\eta=\nabla\alpha=0$

$$\frac{\partial\mathbf{b}}{\partial \mathrm{t}}=\nabla\times(\mathbf{v}\times\mathbf{b})-4\pi\alpha\nabla\times\mathbf{f}$$

$$\frac{\partial\mathbf{v}}{\partial \mathrm{t}}+\boldsymbol{\omega}\times\mathbf{v}=-\frac{\nabla \mathrm{p}}{\mathrm{NM}}-\nabla\left(\tfrac{1}{2}v^2\right)+4\pi\mathrm{C}^2\,\mathbf{f}$$

$$\mathrm{C}^2=\mathrm{B}^2/(4\pi\,\mathrm{NM}),\quad \boldsymbol{\omega}=\nabla\times\mathbf{v}$$

Then $$\frac{\partial\boldsymbol{\omega}}{\partial t}=\nabla\times(\mathbf{v}\times\boldsymbol{\omega})+4\pi\mathrm{C}^2\,\nabla\times\mathbf{f}$$

Hence

$$\frac{\partial}{\partial \mathrm{t}}\left(\mathbf{b}+\frac{\alpha}{\mathrm{C}^2}\boldsymbol{\omega}\right)=\nabla\times\left[\mathbf{v}\times\left(\mathbf{b}+\frac{\alpha}{\mathrm{C}^2}\boldsymbol{\omega}\right)\right]$$

Note that the Hall (vorticity) contribution is smaller O(1/L) compared to the magnetic field. And that makes it the same order as resistive diffusion.

$$\frac{\alpha}{\eta} \sim \Omega_e \tau_e \qquad \Omega_e = \frac{eB}{mc}$$

The Hall effect is a small-scale effect.

See JGR, **101,** 10587-10625, (1996).

If the ion and electron pressures, inertia, and other applied forces $\mathbf{L}_i$, $\mathbf{L}_e$ per unit mass are included, then

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{b}) - \nabla \times [\eta \nabla \times \mathbf{b} - \beta \mathbf{f} \times \mathbf{b} + \alpha \mathbf{f}]$$

$$- \frac{cm/\tau_e}{eBQ} \nabla \times \left[\frac{\nabla p_i}{n} + M\left(\frac{d\mathbf{w}}{dt} - \mathbf{L}_i \right) \right]$$

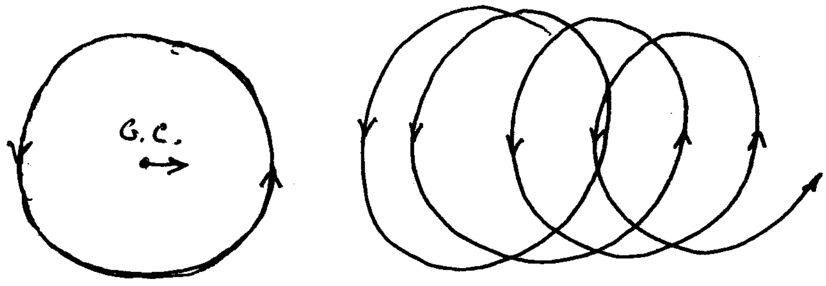
$$+ \frac{cM/\tau_i}{eBQ} \nabla \times \left[\frac{\nabla p_e}{n} + m\left(\frac{d\mathbf{u}}{dt} - \mathbf{L}_e \right) \right]$$

$$- \frac{1}{Q} \nabla \times \left\{ \left[\frac{\nabla (p_i + p_e)}{n} + M\left(\frac{d\mathbf{w}}{dt} - \mathbf{L}_i \right) + m\left(\frac{d\mathbf{u}}{dt} - \mathbf{L}_e \right) \right] \times \mathbf{b} \right\}$$

These extra terms include thermo-electric effects, the Biermann battery, the Eddington-Sweet effect, etc. which are all negligible under ordinary large-scale circumstances in astrophysical settings. But watch out for the small scales arising in tangential discontinuities, rapid reconnection, etc.

Compatibility of Newton and Maxwell

Consider a collisionless plasma, made up of equal numbers of electrons and singly charged ions.
Calculate the electron and ion motions using the guiding center approximation.



Write $\mathbf{u} = c\dfrac{\mathbf{E}\times\mathbf{B}}{\mathrm{B}^2}$, $\qquad \mathbf{E} = -\dfrac{\mathbf{u}\times\mathbf{B}}{c}$

The motion of the guiding center is

$$\mathbf{v} = \mathbf{u} + \frac{\frac{1}{2}\mathrm{M}\,\mathbf{w}_\perp^2 c}{e\mathbf{B}^4}\mathbf{B}\times\nabla\tfrac{1}{2}\mathrm{B}^2 + \frac{\mathrm{M}\,w_\parallel^2 c}{e\mathbf{B}^4}\mathbf{B}\times[(\mathbf{B}\cdot\nabla)\mathbf{B}]$$

Note that

$$\left(\frac{d\mathbf{v}}{dt}\right)_\parallel = -\frac{\mathbf{w}_\perp^2}{2\mathrm{B}^4}\mathbf{B}\{\mathbf{B}\cdot[(\mathbf{B}\cdot\nabla)\mathbf{B}]\}$$

Define

$$p_\perp = \sum_1 \tfrac{1}{2}\mathrm{M}\mathrm{w}_\perp^2, \quad p_\parallel = \sum_1 \mathrm{M}\mathrm{w}_\parallel^2.$$

The current density is

$$\mathbf{j}_{\perp} = \frac{c}{B^2}\mathbf{B}\times\left\{\nabla p_{\perp} - \left[\frac{p_{\parallel} - p_{\perp}}{B^2}\right](\mathbf{B}\cdot\nabla)\mathbf{B} + \mathrm{NM}\frac{d\mathbf{u}}{dt}\right\}$$

and Ampere's law becomes

$$\mathrm{NM}\frac{\mathrm{d}\mathbf{u}}{\mathrm{dt}} = -\nabla_{\perp}\left(p_{\perp} + \frac{\mathrm{B}^2}{8\pi}\right) + \frac{[(\mathbf{B}\cdot\nabla)\mathbf{B}]_{\perp}}{4\pi}\left\{1 + \frac{p_{\perp} - p_{\parallel}}{\mathrm{B}^2/4\pi}\right\}$$

So Ampere's law is automatically satisfied if the bulk velocity **u** satisfies Newton's equation.

See Phys. Rev. **107,** 924 (1957).

Chew-Goldberger-Low Approximation

Let L denote scale of plasma and field in the direction alongthe field. There are, then, four invariants.

$Lw_{\|} = \text{constant}$

$AB = \text{constant}$

$ALN = \text{constant}$

$w^2_{\perp}/B = \text{constant}$

where A is the characteristic cross section of a flux bundle and B the field.

So

$$\frac{d}{dt}\left(\frac{p_{\perp}}{NB}\right) = 0, \qquad \frac{d}{dt}\left(\frac{p_{\|}B}{N^3}\right) = 0$$

$$\frac{dp_{\perp}}{dt} = p_{\perp}\frac{d\ln NB}{dt} + \frac{p_{\|} - p_{\perp}}{\tau}$$

$$\frac{dp_{\|}}{dt} = p_{\|}\frac{d\ln (N^3/B)}{dt} - \frac{p_{\|} - p_{\perp}}{\tau}$$

ELECTRIC CIRCUIT ANALOG

It is asserted that the electric currents required by Ampere's law are subject to the familiar electric circuit equations.

MHD is equivalent to a Laboratory Electric Circuit.

However, in the laboratory circuit:

(a) Current paths have fixed connectivity.

(b) Current paths are fixed in the lab frame.

Whereas in MHD:

(a) Current paths and connections vary according to the dictates of Ampere's law,

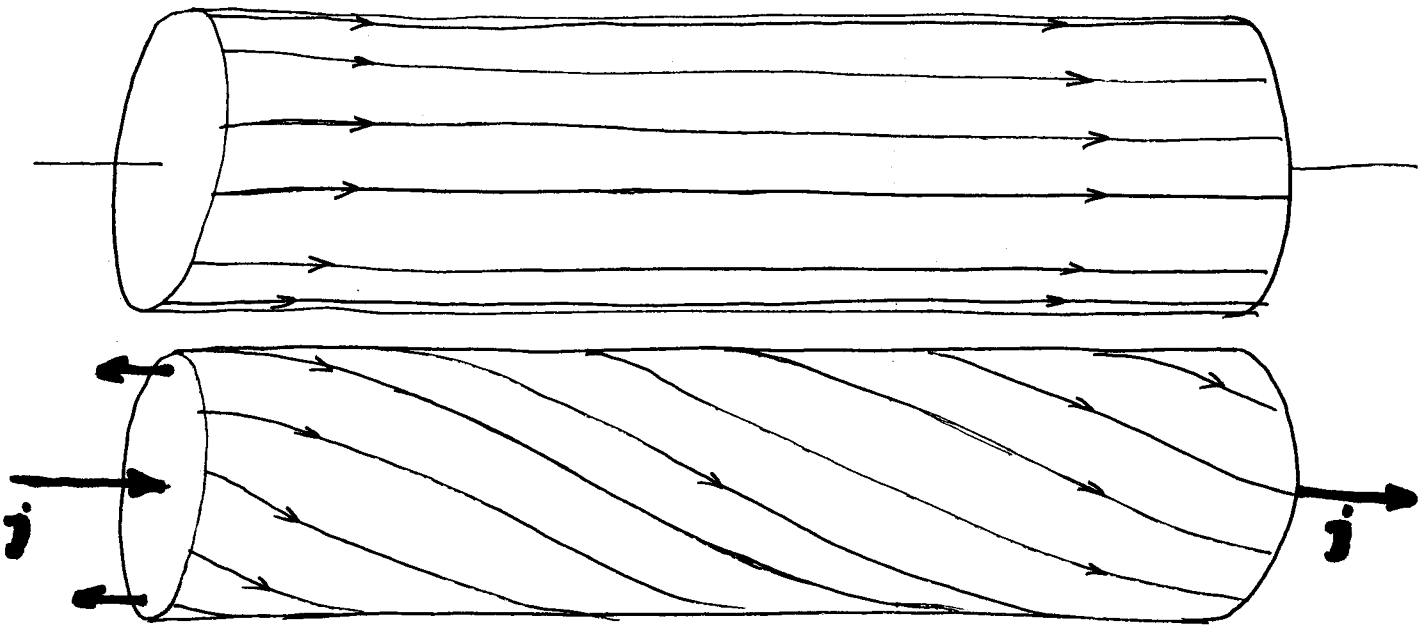
$$4\pi \boldsymbol{j} = c\nabla \times \boldsymbol{B},$$

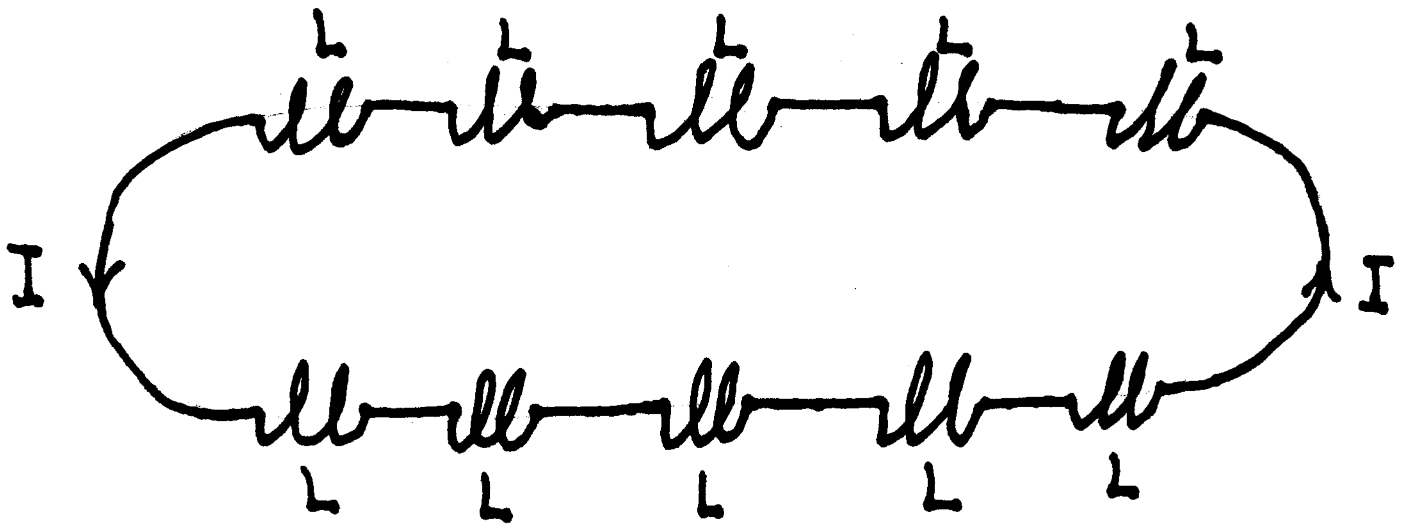
as the swirling fluid velocity $\boldsymbol{v}$ deforms $\boldsymbol{B}$,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}).$$

(b) The current flows in the moving frame of reference of $\boldsymbol{v}$, in which $\boldsymbol{E}' = 0$, so there are no inductive *emf*'s applied to $\boldsymbol{j}$.

There is no electric circuit analog in time-dependent MHD.

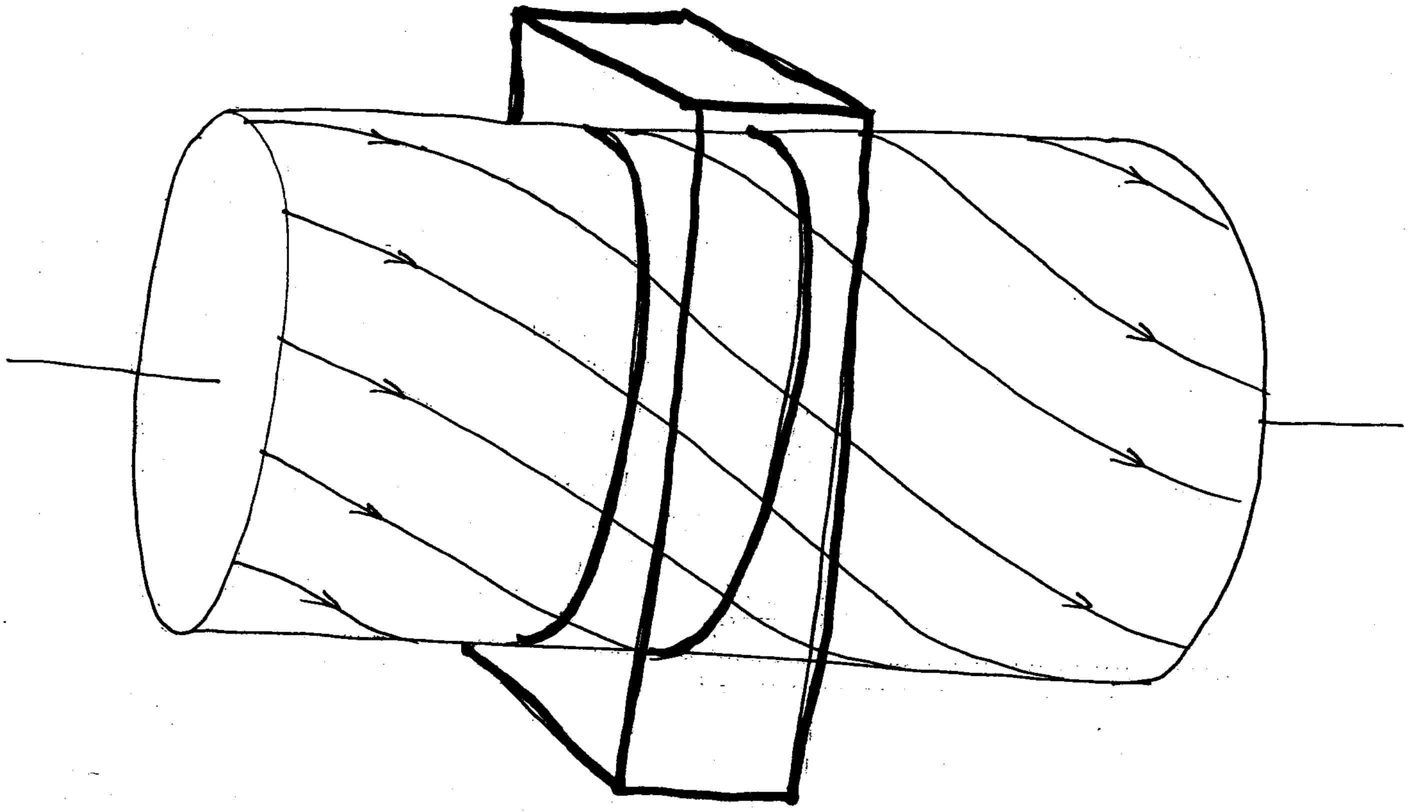


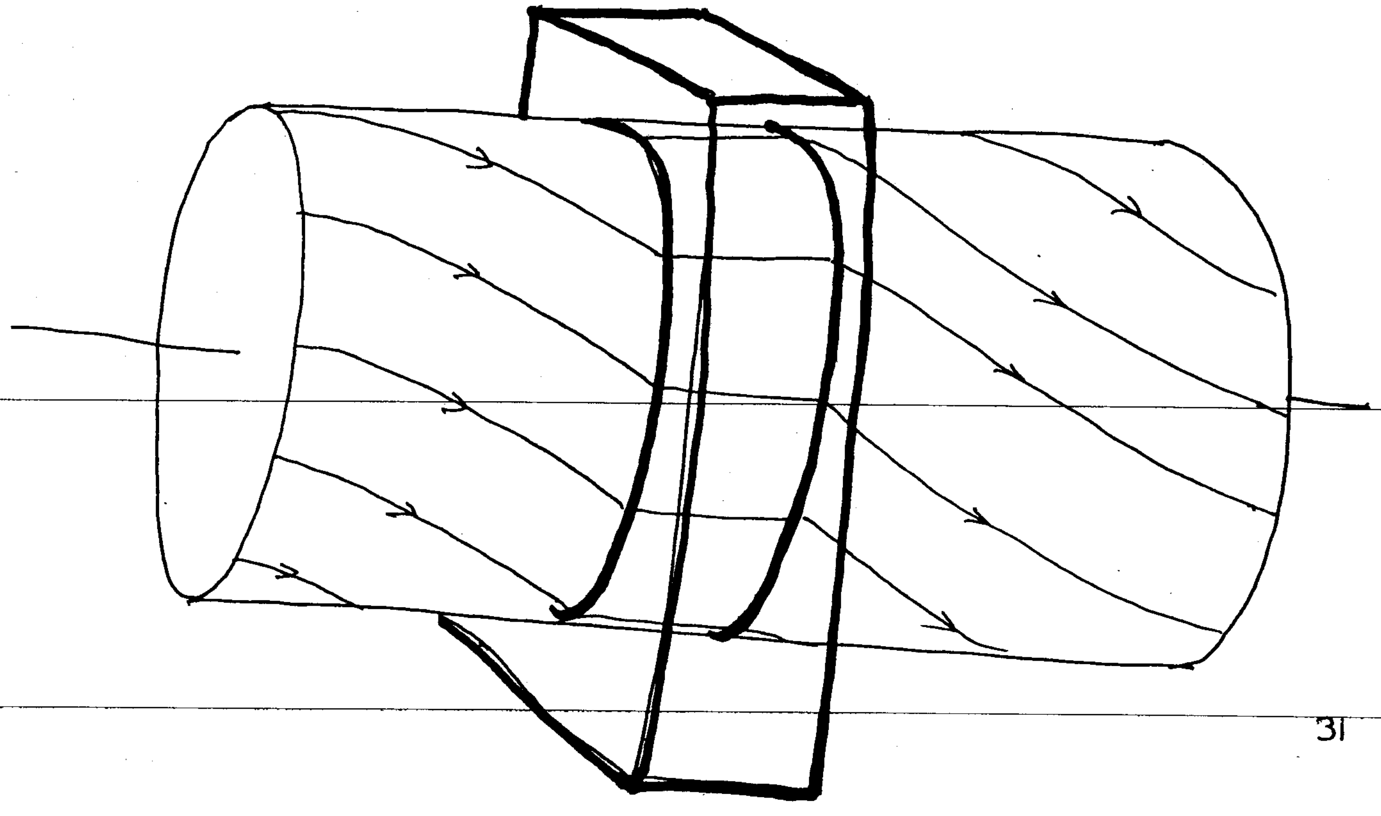


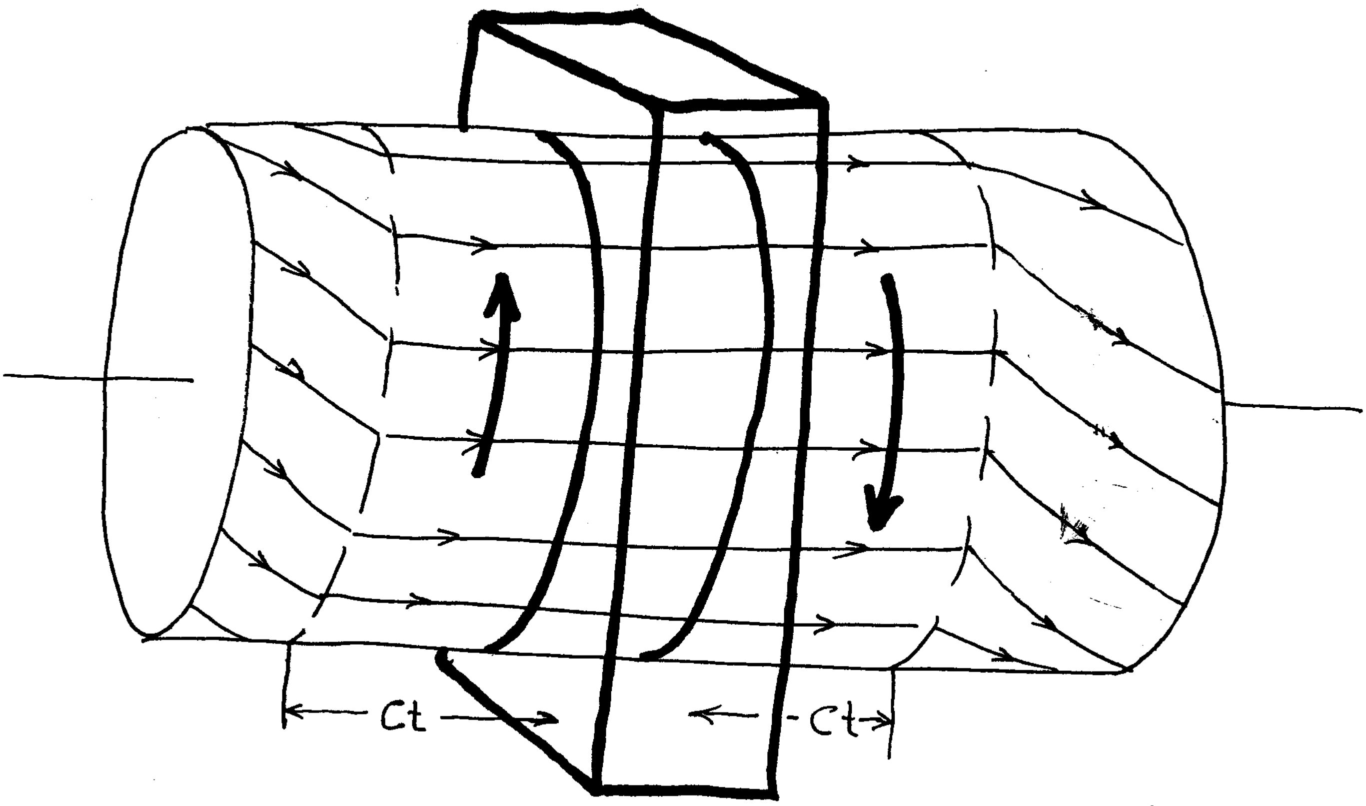
I = Total current

L = Inductance per unit length

$\frac{1}{2}LI^2$ = Magnetic energy per unit length







(c)

Ongoing Developments in Plasma Physics

Laboratory plasma experiments modeling astrophysical effects?

Reinterpretation of stars, galaxies, and the activity of the Sun by laboratory plasma physicists, is sanctified by IEEE Transactions of Plasma Physics.

In their view the Universe is powered by electric currents and voltages, driving the dynamics on all scales and illustrated by numerous laboratory plasma experiments:

Plasma (ionized gas) in magnetic field (usually) driven by application of strong electric potentials (10^5 Volts or more).

Claim to simulate astronomical phenomena, e.g.

Solar Flares

Active coronal filaments

Rotating spiral galaxies (Bostick, Stevens Institute of Technology; Alfven, Stockholm; etc.).

References

Hannes Alfven, 1981, Cosmic Plasma, D. Reidel Publishing Co., Dordrecht.

Donald Scott, 2006, The Electric Sky, Mikamar Publishing, Portland, Oregon.

Donald Scott, 2007, Real Properties of Electromagnetic Fields and Plamsa in the Cosmos, *IEEE Transactions on Plasma Science*, **15**, 822-827.

The assertion is that the Missing Mass Dilemma implies Newtonian theory mechanics and gravity are in error.

And MHD treatment of plasmas is wrong.

Instead, it is proposed that stars and galaxies are basically magnetic structures, activated by electric fields driving electric currents.

The Sun is powered by an electric arc, with $P = VI = 4x10^{26}$ Watts.

Energy sources are not discussed.

Obvious observable side effects are not mentioned.

Centripetal **acceleration** of the Sun in galactic rotation,

$$a_c = \frac{v^2}{r} = \frac{\left(2.5\times10^7\right)^2}{2.5\times10^{22}} \cong \frac{1}{4}\times10^{-7}\,\frac{cm}{\sec^2}$$

Upper **limit of net magnetic** force on Sun,

$$F \approx \pi R^2\,\frac{B^2}{4\pi} \approx \left(0.7\times10^{11}\right)\frac{(10Gauss)^2}{4} \approx 10^{23}\,dynes$$

$$a_B \approx \frac{F}{M} \approx \frac{10^{23}}{2\times10^{33}} \approx 0.5\times10^{-10}\,\frac{cm}{\sec^2}$$

$$\frac{a_B}{a_C} \approx 2\times10^{-3}$$

In fact it follows from the scalar virial equation that the net effect of a magnetic field in three dimensions is expansive. A magnetic field must be forcibly confined if it is not to expand away to infinity. For an isolated "cloud"

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2T + \Phi - M$$

where I is the trace of the moment of inertia tensor, T is the total internal kinetic energy, Φ is the gravitational potential energy (negative), and M is the trace of the Maxwell stress tensor (also negative),

$$M = M_{ii} = -\delta_{ii}\frac{B^2}{8\pi} + \frac{B_j B_j}{4\pi} = -\frac{B^2}{8\pi}.$$

So the idea of magnetic confinement of the stars and gas that make up a galaxy is absurd.

The presumed electric current I passing through the Sun has an enormous magnetic field associated with it. The potential V is not specified, so be generous and assume that V is as large as 10^{10} Volts, requiring that I be only as large as 4×10^{16} Amperes = 1.2×10^{26} esu. Spread out the current to a diameter equal to the radius R of the Sun and compute the characteristic field,

$$B_\phi \approx \frac{2I}{cR} = \frac{2 \times 1.2 \times 10^{26}}{3 \times 10^{10} \times 0.7 \times 10^{11}} \approx 1.2 \times 10^{5}\, Gauss$$

Obaservations show that the general field of the Sun is not more than about 10 Gauss, and with any azimuthal component substantially less.