NO. 72 THE INITIAL REDUCTIONS OF MEASURES ON STAR-TRAILED LUNAR PHOTOGRAPHS

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ABSTRACT

Full details are given for the initial reductions of measures on star-trailed lunar photographs. These reductions determine refraction-free photographic coordinates of limb profile points and points on the disk. These coordinates are referred to axes with the origin at the center of the bright limb with the y-axis directed along the moon's hour circle.

1. Introduction

The long focus refractor is the logical successor to the heliometer for the further development of primary selenodetic research. This paper details the LPL methods for obtaining star-trailed lunar photographs and for making measurements and the initial reductions. These last provide refraction-free photographic coordinates of points on the disk and limb profile in a system oriented on the moon's hour circle.

2. The Star-Trailed Lunar Photographs

During the lunar photographic program at the Yerkes Observatory in the period 1959–1962, star-trailed lunar photographs were attempted on each evening when the moon was more than half full. About 200 star-trailed plates were obtained, but not all of these are expected to be useful. Some are useless because of poor seeing, and on others the trail is too short, being confined to the illuminated portion of the disk. The latter failure comes from using a star that is too faint to register on the unexposed plate (sky or shadow portions), although pre-exposure by the moon's surface sometimes gives a visible trail within the illuminated disk. In the latter case the trail is long enough only when the moon's image is wide in the east-west direction, i.e. near full.

The photographic technique is quite direct but requires both care and quickness on the part of the observer. The finder is first carefully collimated with the telescope, and a decision is made as to which star will be registered as a trail. This star must not be too near the moon or the plate will be fogged during the trail exposure. The lunar exposure is obtained in the normal way and the drive switched off. The plateholder is left open as use of the dark slide is liable to disturb either the holder itself or the plate. In this period lighting in the dome must be kept to a minimum. The telescope is moved a few degrees in declination to intercept the transit of the star. The start and finish of this transit are monitored through the finder. At the end of the transit the dark slide is closed and the plate processed in the usual way.

Using Kodak Contrast Process Ortho plates with the Yerkes 40-in. refractor, we were able to use stars down to the fifth magnitude, although the fainter stars sometimes gave trails only within the bright part of the disk, as already noted.

3. The Measures

The coordinate measures on the star-trailed plates are made with LPL's Mann 422-C comparator. Some details of this instrument are given in *Comm. LPL* No. 61. These measures relate to the trail, to a group of 45 well-defined craters and spots, and to a number of points on the bright limb.

Each plate is measured in two orientations differing approximately by 180°. In each orientation the trail is nearly parallel to the x-axis of the instrument. The measures in the two orientations are united by measures on two fiducial marks. The measures in the second orientation are transformed to the corresponding values in the first orientation by

$$\left.\begin{aligned} x_1 &= x_2 \cos\theta - y_2 \sin\theta + h \\ y_1 &= y_2 \cos\theta + x_2 \sin\theta + k \end{aligned}\right\}. \tag{1}$$

The coefficients θ, h, and k are found as follows: Let (x_1, y_1) and (x_2, y_2) be the readings on the first fiducial mark, and let (x_1', y_1') and (x_2', y_2') be the readings on the second. If we write

$$\begin{aligned} \Delta x_1 &= x_1 - x_1', \\ \Delta y_1 &= y_1 - y_1', \\ \Delta x_2 &= x_2 - x_2', \\ \Delta y_2 &= y_2 - y_2', \end{aligned}$$

then (1) yields

$$\begin{aligned} \Delta x_1 &= \Delta x_2 \cos\theta - \Delta y_2 \sin\theta, \\ \Delta y_1 &= \Delta y_2 \cos\theta + \Delta x_2 \sin\theta. \end{aligned}$$

Solving for $\cos\theta$ and $\sin\theta$, as if these were independents, we get

$$\cos\theta = \frac{\Delta x_1 \Delta x_2 + \Delta y_1 \Delta y_2}{\Delta x_2^2 + \Delta y_2^2},$$

$$\sin\theta = \frac{\Delta x_2 \Delta y_1 - \Delta x_1 \Delta y_2}{\Delta x_2^2 + \Delta y_2^2}.$$

Computed in this way $\sin\theta$ and $\cos\theta$ are not generally consistent, even though the dispersion in the readings on the fiducial marks does not exceed a few microns. Hence, we compute consistent values as follows:

Put $D = \Delta x_1 \Delta x_2 + \Delta y_1 \Delta y_2$,
and $N = \Delta x_2 \Delta y_1 - \Delta x_1 \Delta y_2$.
Then

$$\sin\theta = N/\sqrt{(N^2 + D^2)}, \tag{2}$$
$$\cos\theta = D/\sqrt{(N^2 + D^2)}. \tag{3}$$

The constants h and k are found by applying (1) to both fiducial marks and then averaging, i.e.,

$$2h = (x_1 + x_1') - (x_2 + x_2') \cos\theta + (y_2 + y_2') \sin\theta, \tag{4}$$

$$2k = (y_1 + y_1') - (y_2 + y_2') \cos\theta - (x_2 + x_2') \sin\theta. \tag{5}$$

Note that $\sin\theta$ and $\cos\theta$ follow N and D in sign.

The above formulas are valid for all values of the rotation θ.

The selenodetic points and the trail are measured in both orientations, but the points on the bright limb are observed in the first orientation only. For the selenodetic points, we give the means (x, y) of the first and second orientations and also the semidifferences $(\delta x, \delta y)$. The latter permit the user to assess the reliability of the point.

The star trails are observed at 5-mm intervals, but in general, different points on the trail are observed in the first and second orientations. Hence, it is not possible to determine the precision of the trail measures; in any case, such determinations would be meaningless because of the random deflections due to the seeing. However, an indication of the precision of the direction of the trail is required, and this is discussed below.

4. The Initial Reduction of the Limb Measures

In general, the trail is a short arc of a hyperbola, and the direction of the tangent to this varies from one end to the other. Since our computations for the moon refer to the center of the disk, the appropriate choice of the tangent, as the east-west direction in the sky, is clearly the tangent at the intersection of the trail and the moon's hour circle. In practice this is the foot of the perpendicular from the center of the disk to the trail as in Figure 1. Thus, before we can process the measures on the trail itself,

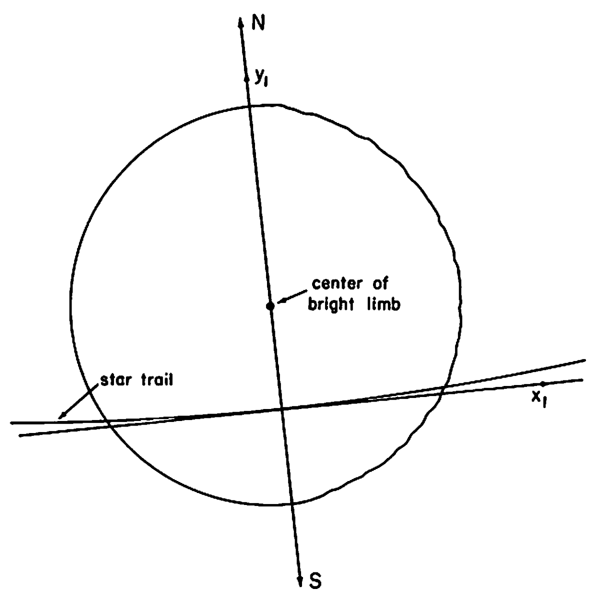

Fig. 1

the approximate coordinates of the center of face must be determined from the measures on the bright limb.

The limb is represented by the circle

$$(x-h)^2+(y-k)^2=r^2. \tag{6}$$

Expanding this and introducing

$$\left.\begin{aligned} g &= r^2-h^2-k^2 \\ z &= x^2+y^2 \end{aligned}\right\}, \tag{7}$$

we obtain the linear observation equation

$$g+2hx+2ky=z \tag{8}$$

for each measured point on the limb. The normals are formed, giving each observation (8) the same weight, and solved for the unknowns g, h, and k. Note that the method of least squares is not correctly applied here, since we minimize the wrong quantity. However, since we are also working with measures affected by refraction, rigor would be wasted at this stage. The approximate values (h, k) are precise enough to determine the center of face for present purposes. The approximate radius of the limb is found from

$$r=+\sqrt{(g+h^2+k^2)}. \tag{9}$$

5. *The Initial Reduction of the Star-Trail Measures*

To a first approximation, the trail is treated as the straight line

$$y=a_1+b_1x.$$

The value of b_1 is found from the normals as

$$b_1=\frac{m\Sigma xy-\Sigma x\Sigma y}{m\Sigma x^2-(\Sigma x)^2}, \tag{10}$$

where m is the number of measured points on the trail. From b_1 we determine the rotation θ_1 which, when applied to the measures, refers these to axes oriented on the trail itself, i.e.,

$$\left.\begin{aligned} \sin\theta_1 &= b_1/\sqrt{(1+b_1^2)} \\ \cos\theta_1 &= 1/\sqrt{(1+b_1^2)} \end{aligned}\right\}. \tag{11}$$

The coordinates of the trail points are transformed by

$$\left.\begin{aligned} x_1 &= x\cos\theta_1+y\sin\theta_1-h' \\ y_1 &= y\cos\theta_1-x\sin\theta_1-k' \end{aligned}\right\}, \tag{12}$$

where

$$\left.\begin{aligned} h' &= h\cos\theta_1+k\sin\theta_1 \\ k' &= (\cos\theta_1\Sigma y-\sin\theta_1\Sigma x)/m \end{aligned}\right\}. \tag{13}$$

This transformation refers the trail points to axes with origin at the intersection of the trail and the hour circle through the center of face. The x_1-axis is directed approximately along the trail.

The star trail is really a hyperbola whose curvature is appreciable when the moon is furthest from the equator. However the trail is so short that we can regard it as circular. Even so, the curvature cannot be reliably estimated from the trail itself, and instead we impose the value

$$\frac{1}{\rho}=\frac{\tan\delta^*}{f}, \tag{14}$$

where f is the approximate focal length of the telescope and δ^* is the approximate declination of the star. Hence, in terms of the transformed coordinates, we write the equation of the trail as

$$y_1=a_2+b_2x_1+c_2x_1^2, \tag{15}$$

where

$$c_2=\frac{\tan\delta^*}{2f}. \tag{16}$$

Writing

$$z_1=y_1-c_2x_1^2, \tag{17}$$

then the typical observation equation is

$$a_2+b_2x_1=z_1,$$

and the least squares solution for b_2 is

$$b_2=\frac{m\Sigma x_1z_1-\Sigma x_1\Sigma z_1}{m\Sigma x_1^2-(\Sigma x_1)^2}. \tag{18}$$

The computations determine b_2 three times, first from the measures in Orientation I, then from the measures in Orientation II as transformed to the system of Orientation I, and lastly from both sets. The first two estimates are used merely to assess the precision of the determination. Only the third value is used in subsequent computations. The quantity b_2, which is small, represents the final rotation from the general direction of the trail to the tangent at the point specified above.

6. *The Correction for Differential Refraction*

The sum $\tan^{-1}b_1+\tan^{-1}b_2$ represents the rotation from the x-axis of the measures to the east-west direction in the star-trail exposure; however, this rotation cannot be taken directly into the lunar exposure because the lunar exposure has a differential refraction which is generally different from that of the trail exposure.

The star's right ascension and declination (α^*, δ^*) are computed manually, and the program then derives the hour angle H^*, zenith distance Z^*, and parallactic angle Q^*, using

$$H^* = \text{Greenwich sidereal time at } 0^h \text{ UT*} + \text{sidereal equivalent of UT*} - \text{W longitude of observatory} - \alpha^*, \quad (19)$$

$$\cos Z^* = \sin\phi \sin\delta^* + \cos\phi \cos\delta^* \cos H^*, \quad (20)$$

$$\sin Q^* = \sin H^* \cos\phi \operatorname{cosec} Z^*, \quad (21)$$

and

$$\cos Q^* = \frac{\sin\phi - \sin\delta^* \cos Z^*}{\cos\delta^* \sin Z^*}. \quad (22)$$

The corresponding quantities for the moon are derived as in *Comm. LPL* No. 60, which also details the computation of the refraction coefficients κ and κ'. Let the values of these for the trail exposure be κ^* and $\kappa^{*\prime}$. We also require the ratios

$$\mu = \kappa/\kappa' \quad (23)$$

for the moon and

$$\mu^* = \kappa^*/\kappa^{*\prime} \quad (24)$$

for the star. Now, in Figure 2 let OW represent the refraction-free direction of the east-west line in both

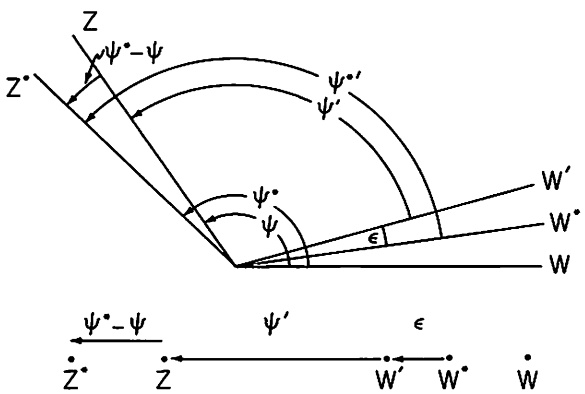

Fig. 2

exposures, and let OZ and OZ^* represent the directions of the zenith in the refraction-free lunar and stellar exposures. Then

$$WOZ = 90° + Q = \psi, \quad (25)$$

and

$$WOZ^* = 90° + Q^* = \psi^*. \quad (26)$$

The refraction compressions operate separately on the two pictures along OZ and OZ^*, thereby leaving these directions undisturbed. Hence, ZOZ^* preserves its value which is $\psi^* - \psi = Q^* - Q$. Let the refraction in the lunar image displace OW to OW', and let the refraction in the stellar exposure displace OW to OW^*. Let $\psi' = W'OZ$ and $\psi^{*\prime} = W^*OZ^*$. Then these last are the values of ψ and ψ^* as affected by refraction, that is,

$$\left.\begin{aligned} \tan\psi' &= \mu \tan\psi \\ \tan\psi^{*\prime} &= \mu^* \tan\psi^* \end{aligned}\right\}. \quad (27)$$

Now let ϵ be the counterclockwise rotation which brings OW^* to OW', that is, the rotation from the apparent east-west direction in the star-trail exposure to the apparent east-west direction in the lunar exposure. It can be seen from Figure 2 that

$$\epsilon + \psi' + (\psi^* - \psi) = \psi^{*\prime},$$

or

$$\epsilon = (\psi^{*\prime} - \psi') - (\psi^* - \psi). \quad (28)$$

7. *The Reduction of the Coordinates to a Refraction-Free Equatorial System*

The final step for the selenodetic points is their transformation to a refraction-free set with the origin at the center of face and the x-axis perpendicular to the moon's hour circle.

It will be remembered that the x-axis of the measures is first rotated through the angle $\tan^{-1}b_1$ to make it approximately parallel to the trail, then through the angle $\tan^{-1}b_2$ to align it precisely along the appropriate tangent to the trail. Lastly, it is rotated through the small angle ϵ to bring it into coincidence with the east-west direction in the lunar photograph, as affected by refraction. This last direction makes an apparent angle ψ' with the vertical in the lunar exposure. Hence, the (x, y) system of measures must be rotated counterclockwise through the angle

$$\xi = \tan^{-1}b_1 + \tan^{-1}b_2 + \epsilon + \psi' - 90° \quad (29)$$

to bring the y-axis into the vertical. Thus for the selenodetic and limb points, the refraction-free horizontal and vertical coordinates are

$$\left.\begin{aligned} u &= \kappa' (x\cos\xi + y\sin\xi) \\ v &= \kappa (y\cos\xi - x\sin\xi) \end{aligned}\right\}. \quad (30)$$

Rotating back through the refraction-free angle Q, the required refraction-free coordinates referred to axes perpendicular to and along the hour circle are

$$\left.\begin{aligned} x_E &= u\cos Q - v\sin Q \\ y_E &= v\cos Q + u\sin Q \end{aligned}\right\}. \quad (31)$$

At this point it must be remembered that the center of face was determined from measures affected by refraction using a method of least squares that was not strictly correct. The computations were precise enough to determine the direction, but not positions, with respect to the limb.

The approximate values h and k are treated exactly like the coordinates of a selenodetic point and reduced by (30) and (31). Let the resulting values be (h_o, k_o). These are small quantities. In the determination of the center of face we seek to minimize the sum of the squares of the radial deviations of the real limb from the circle (6). The deviation of the measured point of the limb from the circle is

$$t = \sqrt{[(x_E - h_o)^2 + (y_E - k_o)^2]} - r,$$

where r is the value found in the first solution. We write

$$R = \sqrt{[(x_E - h_o)^2 + (y_E - k_o)^2]},$$

so that

$$t = R - r.$$

The change in t corresponding to the small changes δh, δk, and δr in h_o, k_o, and r, is

$$\begin{aligned}\delta t &= \delta R - \delta r \\ &= -\frac{(x_E - h_o)}{R} \cdot \delta h - \frac{(y_E - k_o)}{R} \cdot \delta k - \delta \mathrm{r}.\end{aligned}$$

The increment δt is required to cancel the existing discrepancy t, that is, we require $\delta t + t = 0$, from which, approximating R in the denominators to r, our observation equation becomes

$$\frac{(x_E - h_o)}{r} \cdot \delta h + \frac{(y_E - k_o)}{r} \cdot \delta k + \delta r = R - r. \tag{32}$$

The normals are solved for the corrections δh, δk, and δr, which are applied to h_o, k_o, and r. To ensure exhaustion the solution is repeated.

Let (h), (k), and (r) be the final corrected values of h_o, k_o, and r. Then the last reduction is

$$\left.\begin{aligned} x &= x_E - (h) \\ y &= y_E - (k) \end{aligned}\right\}. \tag{33}$$

In these x and y now represent the refraction-free photographic coordinates of the selenodetic and limb points with the origin at the center of the limb circle. The y-axis is directed northwards along the hour circle.

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