No. 75 THE VALIDITY OF SELENODETIC POSITIONS

by D. W. G. Arthur

March 24, 1966

ABSTRACT

Evidence is presented that strong systematic errors in the earthward coordinates of selenodetic points are introduced at all stages of selenodetic triangulation, that is, in the fundamental point determination from heliometer measures, in the determinations of the secondary points, and in the tertiary or photographic triangulations.

In contrast, the best photographic triangulations show virtually no evidence of system in the coordinates parallel to the plane of the limb.

1. Introduction

Several three-dimensional surveys of well-defined points on the lunar surface have been made on the basis of measures on photographs. Here we shall be concerned principally with the Breslau triangulation based on measures by Franz (1901) and reduced by Schrutka-Rechtenstamm (1958), the AMS triangulation (Breece, Hardy, and Marchant 1964), and the ACIC triangulation (Meyer and Ruffin 1965).

All such triangulations depend on a knowledge of the moon's constants of rotation and on a limited number of basic selenodetic positions. The former are used to compute the orientation of the moon at the instant of exposure; the latter are required to connect the arbitrary coordinate system of the measures to the projections of the lunar axes in the plane of the image.

The evolution of the subject and the technical limitations of the last century imposed a hierarchy of basic points. The heliometer method, devised by Bessel for the determination of the constants of rotation, involves determining the selenodetic coordinates of one selected point. This point is connected to a point of the bright limb, in distance and direction, in each heliometer observation. Bessel selected the crater Mösting A for the fundamental point, and it has retained this role ever since in selenodetic work.

Since a single point is insufficient to control either the orientation or the scale of a lunar photograph, Franz (1899) in the period 1890–1894 made a series of heliometer measures at Königsberg connecting eight more points to Mösting A. The correct reductions of these measures give the coordinate steps in the selenodetic coordinate system from Mösting A to these secondary points.

All existing selenodetic coordinates still depend on a position for Mösting A determined from one or another of the heliometer series, and on the Königsberg secondary measures. Whereas several heliometer series are available for the fundamental position, the secondary work of Franz was never repeated, except that Hayn (1904) determined four other secondary positions from micrometric measures at Leipzig. For various reasons these have not been much used as controls for the photographic triangulations.

2. The Fundamental Point

The various heliometer series lead to positions for Mösting A that agree quite well in longitude and latitude, but not quite so consistently in its distance from the so-called center of figure. Variations in this position, in so far as it relates to selenodetic triangulation, are merely equivalent to changes in origin of the coordinates, and are therefore not particularly important. It is merely necessary to ensure that the fundamental and secondary positions are consistent with each other. Thus for the purposes of selenodetic triangulation, any one of the determined positions of Mösting A can be used, but the secondary positions must be consistent with it.

A much different matter exists when we consider the relationship between the selenodetic origin and the lunar limb, or between the origin and the moon's center of mass. It is then important to use the most reliable determination of the position of the fundamental point. In recent years Koziel (1963) has combined four heliometer series into one long series to obtain a position that must be considerably superior to previous determinations based on one series each. Koziel combined the Strassbourg series (1877–1879), the Dorpat series (1884–1885), the first half of the Bamberg series (1890–1912), all observed by Hartwig, and the Kazan series (1910–1915) of Banachiewicz, thus, in effect, obtaining a series extending from 1877 to 1915. The position obtained for Mösting A is

$$\lambda = -5^\circ 9' 50'' \pm 4.5'',$$
$$\beta = -3^\circ 10' 47'' \pm 4.4'',$$
$$\Delta h = +0.40'' \pm 0.19'';$$

this represents the most precise result obtained so far with the methods and data of classical selenodesy. However, we shall see later that there are reasons to doubt the validity of the last coordinate, which is equivalent to an altitude of 0.7 km above the mean spherical datum.

The above position is not used in contemporary work. The secondary measures of Franz and Hayn were reduced by Schrutka-Rechtenstamm (1955), who derived his elements of rotation and position of Mösting A by reworking several heliometer series. The important difference is in the altitude of Mösting A, which Schrutka-Rechtenstamm makes +1.4 km above the mean sphere, that is, twice Koziel's value. The changeover to Koziel's system is not laborious, and should be made by applying constant shifts in all three rectangular selenodetic coordinates of such magnitudes that Mösting A is moved into the position obtained by Koziel.

The standard error for the altitude, viz. 0.19″, is probably quite misleading, as is usually the case in complex observation-reduction schemes in which not all the sources of systematic error are identified and compensated.

3. *The Ranger Radii*

The close-in tracking data for Rangers VI, VII, VIII and IX* permit the calculation of the distances of the impact points from the moon's center of mass. In principle, the calculation is quite simple, requiring only a direct application of the universal law of gravitation to the vehicles' accelerations at impact. Table 1 gives the distance from the impact point to the center of mass in kilometers; the orthographic map coordinates of the impact points are indicated by ξ and η; and K is the selenocentric arc from the center of face.

TABLE 1

THE RANGER RADII

RANGER	DISTANCE (km)	ξ	η	K
VI	1735.3	+0.362	+0.166	23°5
VII	1735.5	−0.349	−0.187	23.3
VIII	1735.2	+0.417	+0.048	24.8
IX	1735.7	−0.040	−0.220	12.9

The dispersion in the distances is surprisingly small, the total spread being only 0.5 km. The mean value, 1735.4 km, is appreciably less than the mean radius, 1738.0 km, of the limb. Even if we allow 1.5 km for the mean difference of elevation between continental surfaces (the limb is almost entirely continental) and the mare surfaces in which the vehicles impacted, there is still a discrepancy of 1.1 km. Either the moon's figure is flattened at the center of face, or the center of mass is 1.1 km nearer to us than the center of figure. At present there is no way of choosing between these alternate hypotheses.

The Ranger radii throw doubts on the accuracy of the selenodetic altitude for the fundamental point Mösting A, whichever hypothesis is adopted. This point cannot lie much more than 0.5 km above the level of the nearby maria — 1.0 km at the most. Thus, if we adopt the idea that the center of mass is nearer than the center of figure by 1.1 km, the distance of Mösting A from the center of mass should not be greater than 1736.4 km, and from the center of figure not more than 1737.5 km. This is 1.2 km below the value obtained by Koziel. On the alternate hypothesis, with a flattening of the moon's figure, the distance of Mösting A should not exceed 1735.4 + 1.0 = 1736.4, which is 1.6 km lower than Koziel's value.

Thus, the Ranger radii are not compatible with the most refined and precise basic selenodetic results. This conflict must be resolved by a critical examination of both sets of data. The Ranger calculations are very direct, whereas the methods of classical selenodesy are well known to be complicated and

*Private communication, D. Trask, Jet Propulsion Laboratory, Pasadena, California.

indirect, involving the orbital theory, the dynamical theory of the moon's rotation, and a difficult application of the calculus of observations. Some minor problems in the rotation theory have not yet been resolved. More important, the heliometer is a complex instrument that is extremely vulnerable to systematic errors. It may be that undetected systematic errors in all the heliometer work are the source of the problem in the height of Mösting A.

4. *The Secondary Points*

Measures for eight secondary positions were made by Franz (1899) in the period 1890–1894 using the Königsberg heliometer. Each point was observed on 12 evenings. The measures have been rigorously reduced by Schrutka-Rechtenstamm (1956). Micrometric measures for four other secondary points were made by Hayn (1904) at Leipzig. The rigorous reductions using modern rotation elements were also the work of Schrutka-Rechtenstamm (1956). These two sets of points are not consistent with each other in their absolute altitudes and earthward selenodetic coordinates.

Table 2 shows the heights in kilometers of the Königsberg points obtained by various authorities. The heights obtained by Schrutka-Rechtenstamm from the Königsburg measures, without and with allowance for phase, are denoted by h_1 and h_1'. The heights obtained by Schrutka-Rechtenstamm (1958) from the Breslau photographic measures are denoted by h_B. Finally, the heights from the ACIC triangulation are indicated by h_C. It is known that the Breslau altitudes are appreciably more precise than the heliometer values, and since

TABLE 2

ABSOLUTE ALTITUDES OF THE KÖNIGSBERG POINTS

POINT	h_1	h_1'	h_B	h_C
Proclus	+ 1.6	+0.3	+0.4	−0.05
Macrobius A	− 1.5	−0.5	+0.1	
Sharp A	−10.2	−8.8	−3.0	
Aristarchus	− 5.4	−8.2	−5.6	
Gassendi	− 5.3	−5.3	−0.1	
Byrgius A	0.0	+2.8	+2.0	
Nicolai A	− 1.3	−0.8	−1.7	+0.02
Janssen K	− 3.0	−2.1	−3.2	−2.73
Mean =	− 3.14	−2.83	−1.39	

$$\Sigma(h_1 - h_B)^2 = 87.1,$$
$$\Sigma(h_1' - h_B)^2 = 72.5,$$

there would appear to be grounds to prefer h_1' to h_1; but the further work of Schrutka-Rechtenstamm is, in fact, based on the coordinates computed without allowance for phase effect.

Table 2 also gives the means of h_1, h_1', and h_B for the eight points. Without phase, the mean altitude of the eight points is 3.14 km below the mean sphere; with phase, 2.83 km below. The application of the Breslau photographic measures reduces this negative mean altitude to 1.4 km. There is clear evidence here that the original heliometer measures are subject to some systematic error that causes the altitudes h_1 to be too low. The same impression results from a comparison with the Hayn secondary points as reduced by Schrutka-Rechtenstamm, which are given in Table 3. The central peak of Tycho, which is quite unsuited for measurement, is omitted.

TABLE 3

THE HAYN POINTS

POINT	HEIGHT (km)
Messier A	−1.5
Kepler A	+0.3
Egede A	−0.8
Mean =	−0.67

The three Hayn points lie in the maria, whereas six of the eight Königsberg points are continental. Hence, the average height of the Hayn points should be less than that of the Königsberg points. In fact, the situation is reversed, and the Hayn points are, on the average, 2.1 to 2.4 km higher than the Königsberg points and 0.7 higher than the Breslau versions of these. Despite the smallness of the samples, it seems safe to assert that there is a systematic difference in height between the Königsberg points and the Hayn points.

Clearly, then, the secondary points are not entirely consistent, and the secondary data are not entirely satisfactory as a basis for the photographic triangulations.

5. *The Breslau Triangulation*

The Breslau triangulation has some importance in contemporary selenodesy. The measures were made by Franz (1901) at Breslau on five Lick plates, and the reductions were made by Schrutka-Rechtenstamm (1958) using his own values of the rotation constants. The triangulation gives curvilinear coordinates, heights, and rectangular selenodetic coordinates of 149 points. The reductions were controlled by the Königsberg points and a single Hayn point that happened to be included by Franz in the Breslau measures. The standard errors listed by Schrutka-Rechtenstamm, which are probably underestimated, show that the coordinates parallel to the plane of the mean limb are known to about 0.1 or 0.2 km. However, the earthward coordinates have random errors of several kilometers.

The Breslau scheme owes its importance to the thoroughness of the reductions and its convenience as a basic net for the control of subsequent triangulations from photographs. Nevertheless, its drawbacks should be noted. It rests on an imperfect secondary scheme. Although the Breslau measures have ironed out the extreme errors of the Königsberg positions, they have not eliminated or even reduced the overall scale, orientation, and datum errors of the secondary net.

6. *Modern Photographic Triangulations*

Recent triangulations of the lunar surface from measures on photographs have been made by Baldwin (1963), the U.S. Army Map Service (Breece, Hardy, and Marchant 1964) and the Aeronautical Chart and Information Center of the U.S. Air Force (Meyer and Ruffin 1965).

The Baldwin triangulation is not completely explicit in terms of coordinates and will not be treated here in detail. The AMS triangulation is of 256 points from measures on 15 photographs. The ACIC initial triangulation of 196 points from measures on photographs taken on eight evenings is the most precise to date. It is the first triangulation based on sequential photography and long-exposure photography, both intended to reduce the seeing distortions. The increased precision of the photography is reflected in the precision of the results.

7. *Some Theoretical Systematic Errors*

Systematic errors in the selenodetic coordinates determined from photographs may originate in many ways. Errors in the control points, random errors in the measures of the controls, errors in the computed librations, systematic errors in the readings of the measuring machine, phase effects, and photographic effects: each of these can produce errors that are functions of position on the disk.

The first three — errors of the controls, random errors of the measures of the controls, and errors of the librations — produce the same type of systematic error. Assume that all points are measured on all photographs, and further, make the simplification of treating the photographs as orthographic pictures. Then the instantaneous coordinates in the plane of the limb are connected to the rectangular selenodetic coordinates by

$$\left.\begin{aligned} X &= aE + bF + cG \\ Y &= eE + fF + gG \end{aligned}\right\}, \qquad (1)$$

in which the coefficients $a, b, \ldots, g$ represent the librations and are subject to the conditions

$$\begin{aligned} a^2 + b^2 + c^2 &= 1, \\ e^2 + f^2 + g^2 &= 1, \\ ae + bf + cg &= 0. \end{aligned}$$

The instantaneous coordinates (X, Y) are connected to the refraction-free photographic coordinates (x, y) by

$$\left.\begin{aligned} X &= px - qy + h \\ Y &= py + qx + k \end{aligned}\right\}, \qquad (2)$$

in which the coefficients p, q, h, k represent the relations between the arbitrary measuring axes and the projected axes of the selenodetic system.

The reductions are now traced to see how the errors influence the final results. First, for a single plate, equation (2), applied to the controls, provides two observation equations for each control point. Forming the normals, the constants p, q, h, and k are found.

With these known for each plate, X and Y can be computed for each point on each plate. Turning now to one point and inserting the values from each plate in (1), we have the observation equations

$$\begin{aligned} a_iE + b_iF + c_iG &= X_i, \\ e_iE + f_iF + g_iG &= Y_i, \end{aligned}$$

each plate yielding one such pair. Forming and solving the normals, the values of E, F, and G are found from

$$\begin{pmatrix} E \\ F \\ G \end{pmatrix} = N^{-1} \begin{pmatrix} \Sigma aX + \Sigma eY \\ \Sigma bX + \Sigma fY \\ \Sigma cX + \Sigma gY \end{pmatrix}, \qquad (3)$$

in which the inverse normal matrix N^{-1} has elements that are functions of the libration coefficients $a_i, b_i, \ldots, g_i$. If there are errors in the instantaneous coordinates X_i, Y_i, the corresponding errors in the selenodetic coordinates are $\delta E, \delta F, \delta G$, where

$$\begin{pmatrix} \delta E \\ \delta F \\ \delta G \end{pmatrix} = N^{-1} \begin{pmatrix} \Sigma a_i\delta X_i + \Sigma e_i\delta Y_i \\ \Sigma b_i\delta X_i + \Sigma f_i\delta Y_i \\ \Sigma c_i\delta X_i + \Sigma g_i\delta Y_i \end{pmatrix}. \qquad (4)$$

Thus $\delta E, \delta F, \delta G$ are linear functions of the errors

δX_i, δY_i for the same point on all plates. Further, from the relations

$$\delta X_i = x_i \delta p_i - y_i \delta q_i + \delta h_i,$$
$$\delta Y_i = y_i \delta p_i + x_i \delta q_i + \delta k_i,$$

it follows that δE, δF, and δG are first-degree functions of the measures (x_i, y_i) for the point on all plates. But from the inverse of (2) and then (1), it follows that δE, δF, δG can be written in the form

$$\left.\begin{aligned} \delta E &= a_{10} + a_{11}E + a_{12}F + a_{13}G \\ \delta F &= a_{20} + a_{21}E + a_{22}F + a_{23}G \\ \delta G &= a_{30} + a_{31}E + a_{32}F + a_{33}G \end{aligned}\right\}, \qquad (5)$$

in which the coefficients a_{ik} are independent of E, F, and G. Thus, whether the errors in E, F, and G arise from errors in the controls or from the measures of these, they are still represented by (5) and are first-degree functions of the selenodetic coordinates.

Errors in the librations lead to errors in p, q, h, and k also, but in addition lead to errors in the matrix N^{-1}. In this case we have the additional errors

$$\begin{pmatrix} \delta E \\ \delta F \\ \delta G \end{pmatrix} = \left(\delta N^{-1} \right) \cdot \begin{pmatrix} \Sigma a_i X_i + \Sigma e_i Y_i \\ \Sigma b_i X_i + \Sigma f_i Y_i \\ \Sigma c_i X_i + \Sigma g_i Y_i \end{pmatrix},$$

where δN^{-1} is the error in N^{-1}. This matrix is again independent of the selenodetic coordinates, and the sums of the two sets of errors in E, F, and G are again represented by (5).

There is little doubt that phase effects produce appreciable systematic errors. The existence of these was noted by Schrutka-Rechtenstamm in his reductions of the Franz and Hayn secondary points. However, I know of no simple way of treating these. Figure 1 shows a bright crater with a thin strip of shadow in the interior. This may be body shadow rather than cast shadow; hence with degradation of the image, it is easy for the observer to place his mark in the center of the bright area rather than the center of the crater. Obviously the chance of an error of this kind is increased if the picture lacks resolution and the observer lacks alertness and experience. The phase effect displaces the crater toward the terminator.

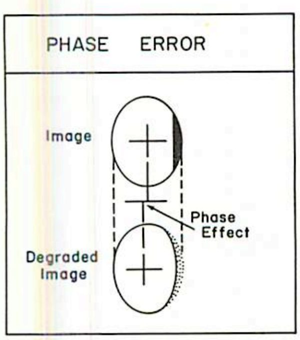

Fig. 1

Photographic effects resemble phase effects when measures are made on the original negatives, in which case the dark areas within the crater and around the crater rim are intruded on and diminished by the bright areas. The net result is a displacement of the crater image toward the terminator as indicated in Figure 2. However, this is countered to some extent by irradiation in the eye of the measurer.

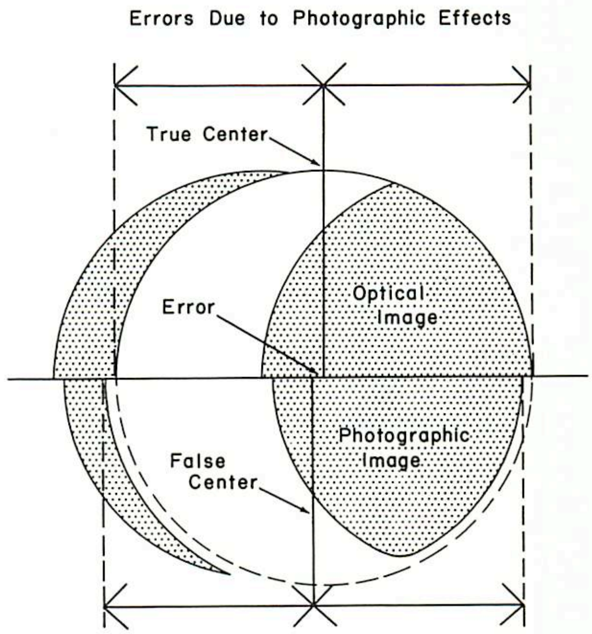

Fig. 2

The effects of these errors on the coordinates depend on the statistical relations between the librations and phase. The results of phase and photographic effects will not be discussed further here, but they should certainly be kept in mind because, as will emerge below, the selenodetic coordinates are affected by systematic errors that do not come from the sources treated analytically above, namely, the controls and their measures or the librations.

8. *The Height Regressions*

The most conspicuous common feature of the Breslau, AMS, and ACIC triangulations is the trend toward large positive heights at the center of face and either zero heights or negative heights near the limb.

It would be possible to analyze the height as a statistical function of position on the disk, but in order to deal successfully with the large dispersion, it is best to reduce the number of parameters to a minimum. Therefore, we follow Baldwin and consider the height as a function of one variable only, namely the distance from the center of face. Let K be the selenocentric arc measured from the center of face. Then we can choose one or another of K, $\sin K$, $\cos K$, or even some other function of K as our argument. Obviously, the preferred argument is the one that gives a straight line regression for the height.

Returning to (5), since

$$\delta h \approx E\delta E + F\delta F + G\delta G,$$

the expression for the systematic error in h takes the form

$$\left.\begin{aligned}\delta h = \quad & b_1E + b_2F + b_3G \\ & + b_{11}E^2 + b_{22}F^2 + b_{33}G^2 \\ & + b_{12}EF + b_{13}EG + b_{23}FG\end{aligned}\right\}. \qquad (6)$$

This represents the systematic height error due to errors in the controls and the librations. It contains nine coefficients, far too many in view of the large dispersion in the heights. Hence, we average out the trends along the various directions from the center of face as follows: First we introduce the approximate substitutions,

$$E = \sin K \cos P,$$
$$F = \sin K \sin P,$$
$$G = \cos K,$$

where P is a position angle about the center of face. Introducing these in (6) and replacing the functions of P by their average values, we are left with

$$\left.\begin{aligned}\delta h &= a + b\cos K + c\cos^2 K \\ &= a + bG + cG^2\end{aligned}\right\}. \qquad (7)$$

We can also take the effects of lunar figure into account in the same way. If the figure is ellipsoidal, then averaging out the characteristics of the great circles through the center of face, we have

$$h = \beta \cos^2 K = \beta G^2. \qquad (8)$$

Thus, whether the height comes from an ellipsoidal figure, from errors in the controls or their measures, from errors of the librations, or from all of these, we would expect a regression represented by (7), i. e.,

$$h = a + bG + cG^2. \qquad (9)$$

The data for the three triangulations can indeed be fitted by (9), but the coefficients indicate that they can be fitted in a simpler way by

$$h = A - B\sin K \qquad (10)$$

for all three triangulations. This result is quite unexpected and to some extent unwelcome, since there are serious difficulties in the interpretation of (10).

The least squares analysis is limited to continental points, since there are reasons to believe that the maria are depressions and not typical of the moon's figure. The results for the three triangulations are given in Table 4. The heights are in kilometers.

TABLE 4
THE HEIGHT REGRESSIONS

POINTS	BRESLAU 46	AMS 130	ACIC 19
σ	± 1.25 km	± 1.78 km	± 0.82 km
A	$+3.82 \pm 0.48$	$+5.00 \pm 0.55$	$+2.27 \pm 0.54$
B	$+6.01 \pm 0.72$	$+4.95 \pm 0.71$	$+2.70 \pm 0.82$

Whatever the nature of the regressions, it is clear from the values for the standard deviation σ from the regression line (corrected for 2 degrees of freedom) that the ACIC triangulation has the smallest random errors and the AMS the largest. The value of 0.82 for the ACIC triangulation is surprisingly small, for this figure includes the effects of topography and the variations in the characteristics of the different radials. Even though the use of (10) does not imply that the lunar figure is one of revolution about the moon's first radius, the regressions show that it cannot differ much from such a figure. The regressions for the Breslau, AMS, and ACIC triangulations are shown in Figures 3, 4, and 5.

9. *The Interpretation of the Regressions*

The regressions represent either the moon's figure or systematic height errors, or a combination of the two. If the moon is strictly spherical, then (10) represents only the systematic height error.

It is easily shown that (10) cannot represent only the figure. The lunar limb is entirely continental except for the short arcs occupied by the limb maria. Its mean radius is 1738.0 km; hence, we would expect the mean altitude of continental points near the limb to be slightly greater than 1738.0 km — slightly greater because the limb maria can be expected to lower the mean radius by a small fraction of a kilometer. Thus if (10) represents real heights and nothing else, then the regression should converge on a small positive value as K goes to 90°. That is, we should have

$$A - B = \epsilon \qquad (11)$$

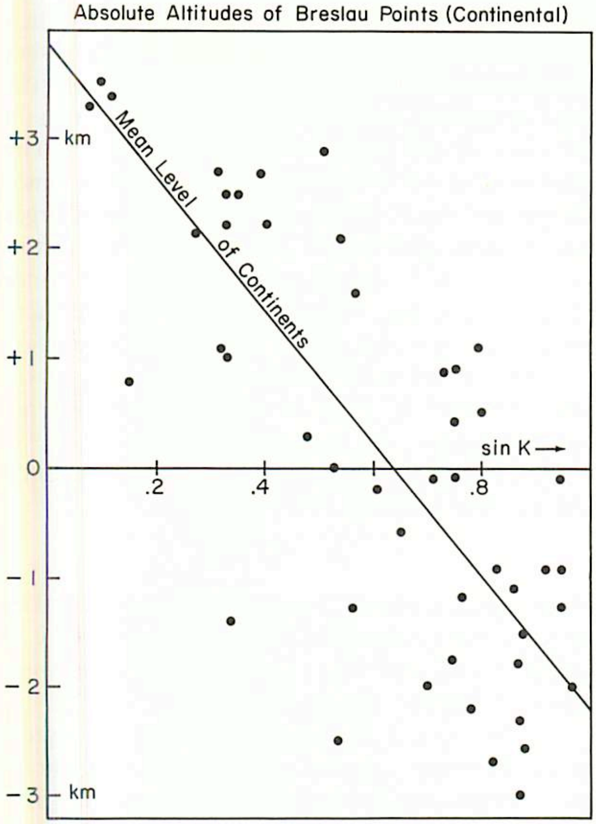

Fig. 3

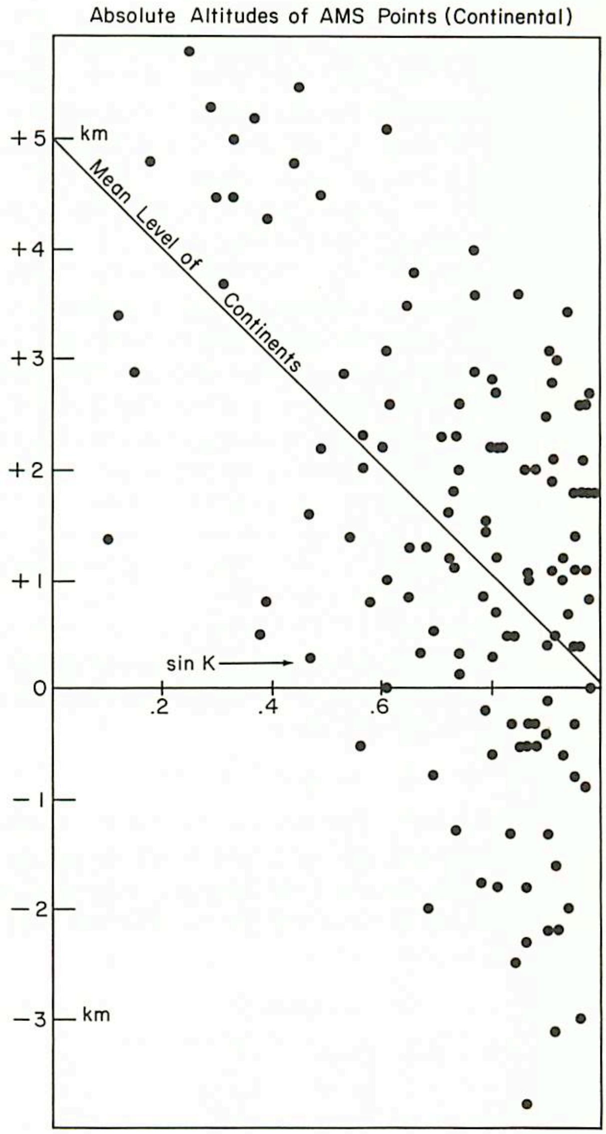

Fig. 4

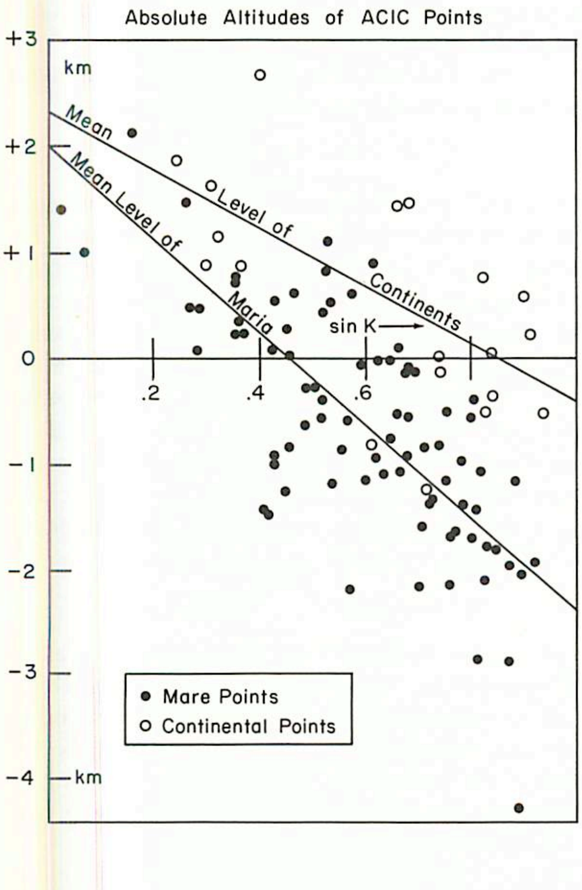

Fig. 5

where ϵ is positive and small, say less than 0.2 km. However, the values of ϵ for the Breslau, AMS, and ACIC triangulations are -2.19, $+0.05$ and -0.43 km respectively. The Breslau triangulation fails to meet the criterion, and there is some doubt about the ACIC triangulation.

Since the regression for the Breslau net represents systematic error at least in part, either the figure terms are linear in sin K, or the nonlinear terms in the systematic error and the figure balance out exactly. The latter is so improbable that it must be rejected. The other alternative, that the figure terms are linear in sin K, is also hardly credible, since $A - B \sin K$ is equivalent to a rather strange figure with a conical boss at the center of face. It is much easier to believe that (10) represents systematic height error and nothing else.

If we accept this, then the conclusion also applies to the other two triangulations, for the regressions are of exactly the same type. Thus the form of (10) brings forth the discouraging conclusion that none of the three triangulations give firm evidence of deviation from the spherical form and that one of them has systematic height errors ranging up to +5 km.

There are even more puzzling features in the form of (10). There is no evidence of terms in cos K and $\cos^2 K$, and hence, no indication of systematic errors from sources that we know are present.

The validity of (10) must therefore be carefully examined because it leads to a rather puzzling situation. To confirm the linearity in sin K, the same data were fitted to the expression

$$h = A' - B' \sin K + C' \sin^2 K. \qquad (12)$$

If this is a better representation of the data, the dispersion about the regression line should be smaller. The results for the three triangulations are shown in Table 5. It will be noted that the standard deviations

TABLE 5
NONLINEAR HEIGHT REGRESSIONS

POINTS	BRESLAU 46	AMS 130	ACIC 19
σ	+1.25 km	+1.78 km	+0.84 km
A′	4.42	5.74	2.95
B′	8.78	7.84	5.50
C′	2.48	2.36	2.38

about the regression line are exactly the same. From the viewpoint of dispersion, the linear and quadratic regressions have equal validity. However, the values of A and B are made larger and, in my view, much less probable. I would reject the quadratic expression completely but for one surprising feature, namely, the close agreement between all three triangulations in the value of the second-degree coefficient C'. The value of C' is +2.4 with a spread of 0.12 km. Since the standard error of C' in each triangulation is larger than 10, this agreement is remarkable. If not fortuitous, it implies a spheroidal flattening at the center of face of 2.4 km, which however is overlaid and completely concealed by the non-spheroidal elongation represented by A and B. Certainly, any curvature concealed in (10) is not consistent with an earthward spheroidal elongation of the figure.

The quadratic regression (11) can also be tested with the limb criterion. This is now

$$A' - B' + C' = \epsilon,$$

where again ϵ should be small and positive. The Breslau, AMS, and ACIC triangulations give $\epsilon = -1.88$, $+0.26$, and -0.17 km respectively, and once again the Breslau triangulation fails.

At this stage I am reluctant to reject the regression (12) completely, but the Baldwin triangulation gives a regression for the continental points in which C' is about -6.0. If we accept these data, the conclusion must be that the agreement is fortuitous and that C' has no significance for the figure.

In the Baldwin triangulation, the continental and mare points give regressions of the type

$$h = A' - B' \sin K + C' \sin^2 K$$

in which C' approximates to -6.0 in the continental regression and $+6.0$ in the mare regression. The average of the two regressions is therefore represented by (10). However, since the mare regression rises above the continental regression at the center of face, there is something rather strange in the results. Evidently, the regressions are not valid representations of the heights close to the center of face. This triangulation, like the others, tends to negative heights near the limb. Clearly there are appreciable systematic errors in Baldwin's results. His continental and mare regressions are shown in Figure 6.

10. Systematic Scale Differences

Even for contemporary triangulations, scale is still imposed by the Königsberg measures of 1890–1896. Unfortunately, these measures were published in a much reduced form, and there are no details concerning the conversion to arc or the removal of refraction. By 1890 however, the Königsberg heli-

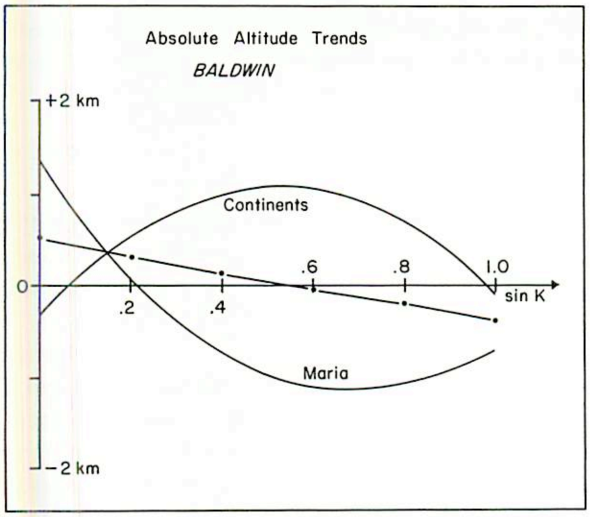

Fig. 6

ometer was an instrument with well-known characteristics and had been repeatedly calibrated. There is no reason to believe that sensible systematic scale errors were introduced in the measures of the Königsberg points.

Despite this, the notes of Franz (1899) indicate a curious discrepancy that appears to have been passed over by Schrutka-Rechtenstamm. The scales of the five Lick plates measured at Breslau were controlled by the Königsberg points. Franz also measured seven points on the profile of the bright limb of each plate, so that he was able to derive a theoretical radius of the limb from the Königsberg points, and also what we might call an observed radius from the seven points. Table 6, taken directly from

TABLE 6

THEORETICAL AND OBSERVED RADII
BRESLAU PLATES

PLATE	OBSERVED − THEORETICAL
I	+1″55
II	+1.30
III	+0.73
IV	+1.06
V	−0.11

the results given by Franz, shows the excess of the observed value over the theoretical value. The average excess is almost one second of arc. Franz attributed this to irradiation, but the anomalous value for the last plate shows that lack of rigor in the Franz reductions of the secondary points may have played some role too.

Certainly, in the reductions of the Yerkes plates, using the Breslau points as controls, there is little evidence of irradiation. The observed radii, derived from at least 30 limb points, tend to exceed the theoretical radii, but the amounts are only one tenth of those found by Franz. Our measures thus confirm that the scale error of the Breslau net cannot exceed 0.0002, and may be much smaller.

The AMS and ACIC triangulations, which are based on the Breslau triangulation, may be tested against it for scale. Since I believe that the heights, and thus the earthward coordinates G, are seriously affected by systematic errors, the test is restricted to scale parallel to the plane of the mean limb. Let

$$R = \sqrt{(E^2 + F^2)} \tag{13}$$

where E and F are the rectangular selenodetic coordinates in the plane of the limb, that is, the ξ and η of Schrutka-Rechtenstamm. Then, between two triangulations we have

$$\left.\begin{aligned} R' &= (1+\mu)R \\ \text{or}\quad \Delta R &= \mu R \end{aligned}\right\}. \tag{14}$$

The least squares value of the scale difference is

$$\mu = \Sigma R \Delta R / \Sigma R^2, \tag{15}$$

and the variance of μ is

$$\sigma_\mu{}^2 = \sigma^2 / \Sigma R. \tag{16}$$

The standard error σ of a single scale comparison is found from

$$\sigma^2 = \frac{\Sigma v^2}{n-1} = \frac{\Sigma(\Delta R - \mu R)^2}{n-1}. \tag{17}$$

The above can first be applied to a comparison of the Königsberg and Breslau values, both as computed by Schrutka-Rechtenstamm. The data are given in Table 7. The table gives

$$\mu = +0.00019 \pm 0.00004.$$

The difference is selenodetically significant and per-

TABLE 7

SCALE DIFFERENCES, BRESLAU — KÖNIGSBERG

POINT	R	ΔR
Proclus	+0.75436	+0.00030
Macrobius A	.69570	+ .00022
Sharp A	.86712	+ .00026
Aristarchus	.78548	+ .00009
Gassendi	.71054	− .00020
Byrgius A	.91552	+ .00049
Nicolai A	.73633	− .00003
Janssen K	.85740	.00000

haps larger than anticipated, though this difference could easily arise from rejections of some of the points in the reductions of the plates.

The distribution of the common points between the Breslau and ACIC triangulations, at different values of R, permits an investigation of ΔR as a function of R. Apparently, for the difference ACIC − Breslau, ΔR follows a third-degree curve, starting from zero at $R = 0$, reaching a maximum negative of −0.00007 at $R = 0.28$, then returning to zero at $R = 0.56$, and finally climbing to +0.00009 at $R = 1.0$. The value of R found from points in the zone $0.3 \leqslant R \leqslant 0.6$ is -0.00009 ± 0.00004, and in the zone $0.6 \leqslant R \leqslant 1.0$ is $+0.00009 \pm 0.00004$. The overall value is +0.00005. Thus, although the overall scale difference between ACIC and Breslau is almost negligible, there is some evidence that the values differ systematically in a nonlinear fashion.

The comparison between Breslau and AMS indicates a relatively large scale difference. From the difference R(AMS) − R(Breslau) for the 61 common points, we have

$$\mu = +0.00068 \pm 0.00005.$$

Indeed, ΔR is negative at one point only. The regression of ΔR on R is shown in Figure 7. In view of the

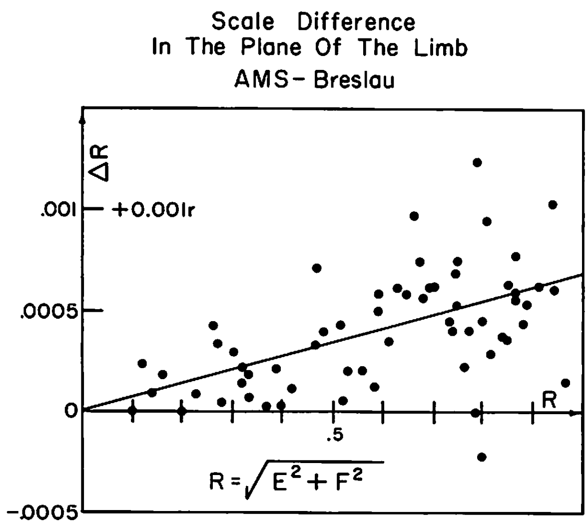

Fig. 7

close agreement in scale among Breslau, AMS, and the photographic measures at LPL, it is evident that the AMS triangulation contains serious scale error amounting to 1 km at the limb. Since the regressions also show that this triangulation has an exceptionally large non-ellipsoidal elongation, and high dispersion, it is clear that there is some defect in the AMS work that requires attention before this triangulation is extended.

11. Systematic Differences of Level between Continents and Maria

In the present imperfect state of selenodesy, the relative levels of maria and continents have a temporary importance. Goudas (1966) recently asserted that the selenodetic results do not support the traditional view that the maria are lower than the continents. However, this idea cannot have come from a careful interpretation of the data. In the present investigation, the mare points were not taken into the least squares analysis of the regressions discussed in Section 8, but were plotted. For all three triangulations it was evident that the bulk of the mare points fell below the continental regression lines, the average depression of the mare points being about 1.25 km below the regression lines. The interpretation of the regressions is quite irrelevant in this context and the result holds whether they represent real heights or systematic errors. Figure 8 shows the relative levels of

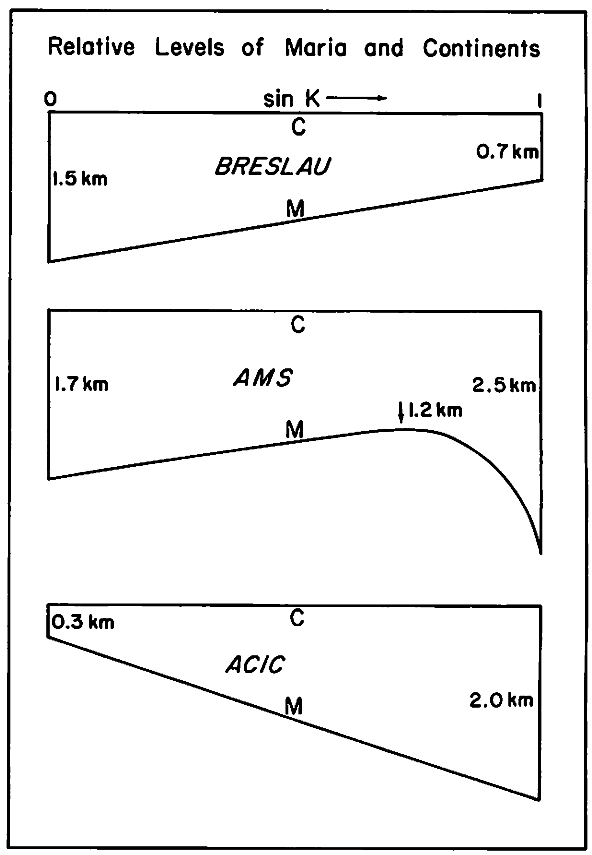

Fig. 8

continental surfaces (C) and mare surfaces (M) at various values of sin K. Note the peculiar dip in the AMS mare levels in the areas near the limb. The difference is smaller than anticipated, but this may be due to bias in the sample. The selenodetic points tend to avoid deep maria such as Crisium.

That the maria are depressed below the continents is also easily seen from the limb profiles. From the LPL measures of such profiles, it is clear that the central regions of Smythii, Marginis, and Humboldtianum lie 5, 2, and 4 km respectively below the mean level of the limb. Weimer's charts (1952) indicate that the central area of Orientale lies about 3 km below the mean level. The mean central depression for the four limb maria is thus 3.5 km, that is, the average depression of their surfaces is about 1.75 km. Combining the two sets of data, the overall mean depression for mare surfaces is about 1.5 km. This figure should be used with some caution since the lunar maria are distinct individuals with widely different depths.

12. Deformations of the Model

The expression

$$\delta h = A - B \sin K,$$

regarded as a representation of the systematic errors, is not explicit as to the manner in which these deform the model of the moon. In conformity with the mode of analysis used here, we use only two coordinates, R and G. We then have

$$R\delta R + G\delta G = A - BR.$$

This has an infinite number of solutions for δR and δG. For the AMS and ACIC triangulations, errors in R seem to be too small to generate appreciable height errors, and it would appear that for these two triangulations, the height regressions are due entirely to systematic errors in G described by

$$\delta G = (A - BR)/G.$$

This indeed may be the case, but it must be remarked that the *differences* between these two triangulations do *not* support this conclusion. From Table 4, we obtain for ACIC − Breslau

$$\Delta G = -(1.55 - 3.31R)/G,$$

whereas when $G\Delta G$ is plotted against R, there is no sign of a regression. This discrepancy may arise from the fact that 30 of the 41 common points are used as controls by ACIC. Moreover, there is a shift in the sampling. The regressions are based on all continental points in the two triangulations, but no distinction is made between mare and continent in the common points used to evaluate $G\Delta G$.

Despite this contradiction, it is clear that the height difference Δh must come from differences in G in the two triangulations, and indeed, the correlation coefficient for Δh and $G\Delta G$ is greater than $+0.9$.

The above should make clear some of the difficulties in using the regressions to explore the nature of the systematic errors. With the dispersions present in this case, the number of points is too small. The latter difficulty is aggravated in the direct comparison of two triangulations, and such a comparison between the AMS and ACIC results is hardly worth while because of the paucity of common points.

The form $\delta h = A - B \sin K$ for the height error is rather puzzling, since it rules out as sources errors of the controls or their measures, errors of the librations, and departure of the center of figure from the origin of coordinates. Nor can the regression be attributed to an ellipsoidal figure. For the present I must leave the solution of this problem and the confirmation of the validity of the linear form $A - B$ sin K to the arrival of more and better data. In addition, it should be noted that the regressions of Saunder (1905), Weimer (1954), and Baldwin (1963) are all strongly curved. The regressions for all six triangulations are shown in Figure 9.

13. Conclusions

The above should make clear that to date we have achieved only partial success in one of the principal objectives of selenodesy, namely, the determination of reasonably precise selenodetic coordinates. The coordinates (E, F) parallel to the plane of the mean limb appear to be reasonably well determined in the fundamental, secondary, and photographic triangulations. The earthward coordinate G is not well determined. There is reason to believe that the best heliometer value of G for Mösting A is still not correct and that all heliometer values for this quantity are too high.

The secondary triangulations of Franz and Hayn show a number of unsatisfactory features, but there is no long range problem here as work now approaching completion at LPL will provide a new and more precise secondary net.

The photographic or tertiary selenodetic triangulations again display large errors in G, though the precision in E and F is quite satisfactory. In part this

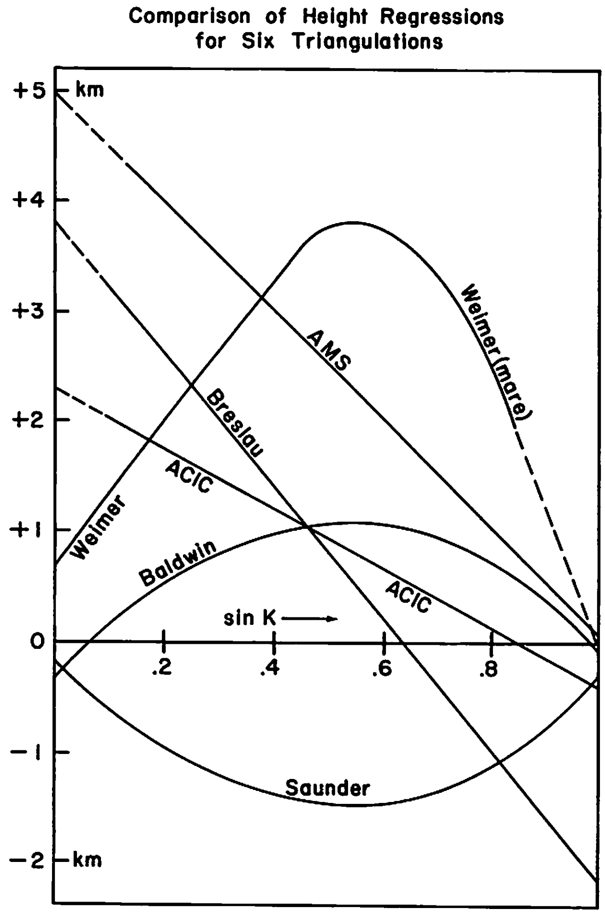

Fig. 9

is due to the geometry of the situation, but this does not explain the systematic errors encountered in all photographic triangulations. Many further measures of the type made by ACIC are required to resolve this problem, since as the measures accumulate, the random errors of the selenodetic positions diminish, and the character of the systematic errors becomes clearer.

Furthermore, even strong systematic position errors are not in themselves objectionable, so long as they are truly systematic, and the law of the system eventually becomes available.

The precise coordination of the lunar surface points still presents considerable problems. These must be met by more and better measures, and by a more rigorous and refined analysis of the results.

Acknowledgments. Financial support for this research was provided by the Air Force Cambridge Research Laboratories, U.S. Air Force, under Contract AF19(628)–4332, and by the National Aeronautics and Space Administration under Grant No. NsG 161-61.

REFERENCES

Baldwin, R. B. 1963, *The Measure of the Moon* (Chicago: University of Chicago Press), pp. 212–245.

Breece, S., Hardy, M., and Marchant, M. Q. 1964, *Horizontal and Vertical Control for Lunar Mapping,* Army Map Service Tech. Report No. 29, Pt. II.

Franz, J. 1899, "Die Figur des Mondes," *Astr. Beob., Königsberg,* Vol. 38.

————. 1901, "Ortsbestimmung von 150 Mondkratern," *Mitt. Sternwarte Breslau,* I.

Goudas, C. L. 1966, "Note on 'Shape and Structure of the Moon' by Lamar and McGann," *Icarus,* 5, 99–101.

Hayn, F. 1904, *Selenographische Koordinaten* (Leipzig: Sächs. Gesell. der Wiss.), II Abh.

Koziel, K. 1963, "Crater Mösting A as the First-Order Point of Triangulation on the Moon," *Fourth International Space Science Symposium, Warsaw, 1963.*

Meyer, D. L., and Ruffin, B. W. 1965, *Coordinates of Lunar Features, Group I and II Solutions,* ACIC Tech. Paper No. 15, Aeronautical Chart and Information Center, U.S. Air Force, St. Louis, Mo.

Saunder, S. A. 1905, "First Attempt to Determine the Figure of the Moon," *M. N.,* Vol. 65, No. 5.

Schrutka-Rechtenstamm, G. 1955, "Zur Physischen Libration des Mondes," *Sitz. Österr. Akad. Wiss. Math.-Naturw. Kl.,* Abt. II, 164 Bd.

————. 1956, "Neureduktion der acht von J. Franz und der vier von F. Hayn gemessenen Mondkraterpositionen," *Sitz. Österr. Akad. Wiss. Math.-Naturw. Kl.,* Abt. II, 165 Bd.

————. 1958, "Neureduktion der 150 Mondpunkte der Breslauer Messungen von J. Franz," *Sitz. Österr. Akad. Wiss. Math.-Naturw. Kl.,* Abt. II, 167 Bd.

Weimer, T. 1952, *Atlas de Profils Lunaires* (Paris: Observatoire de Paris).

————. 1954, "Recherches sélénographiques: Allongement du sélénoid; Libration physique; Profils lunaires," *Bull. Astr.,* Tome XVII, Fascicule IV.