No. 76 A METHOD FOR DETERMINING THE MOON'S CONSTANTS OF ROTATION FROM MEASUREMENTS ON SCALED AND ORIENTED LUNAR PHOTOGRAPHS

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ABSTRACT

This paper describes a method for determining the moon's constants of rotation and the selenodetic coordinates of a number of primary points starting from rectangular coordinate measures on scaled and oriented lunar photographs. The new features are the use of Mösting A as the origin of selenodetic coordinates and the employment of rigorous formulas in place of the approximate differential formulas of classical selenodesy. The least squares analysis takes into account the algebraic correlation introduced in the reductions of the observed points.

1. *Introduction*

The constants of the moon's rotation, namely the ratios of the moments of inertia, $\beta = (C - A)/B$ and $\gamma = (B - A)/C$, have in the past always been determined from heliometer measures that connect the fundamental point (Mösting A) to the bright limb, according to the scheme set out by Bessel in 1839. Technically there has been relatively little advance since Bessel's time, except for a more thorough working out of the reduction scheme by Hayn (1902), and a rigorous application of the calculus of observations by Koziel (1962). At present, the best values we have for β and γ come from four relatively elderly heliometer series which were combined into a single long series by Koziel (1963).

When all possible credit has been given to those responsible for these numerous heliometer measures and their reductions, it must still be remarked that the heliometer technique belongs to the last century. It is a pre-photographic method with important limitations in both measurement and reduction. The measures are slow, and the net results of an evening's work are only seven to ten distances from the fundamental point to the bright limb: these have to be reduced to a single instant. A photograph containing the same information can be obtained in a few seconds.

However, there is another limitation just as important. The Bessel-Wichmann scheme for reducing the measures involves assumptions about the nature of the lunar limb. The bright limb, smoothed to a circle, is supposed to define an invariant point in the moon's interior, the so-called center of figure. When the measures attain a certain level of precision, this concept is not valid, since the center of figure is then a blurred region whose dimensions are of the same order as the errors of the measures. Later refinements, such as those by Yakovkin, do not really surmount the difficulty.

It is of interest, of course, to relate the selenodetic coordinate system to the moon's centroid; if this is required, assumptions of some sort concerning the limb and its relation to the centroid become necessary. Nevertheless, as shown below, the determination of the constants of rotation and of differential selenodetic coordinates can be divorced completely from relations between the limb and the centroid. An approach of this type is essential if earth-based selenodesy is to progress beyond the limitations set by the heliometer method. The new method must come from the use of photographs that permit a large number of effectively simultaneous differential measures between well-defined spots on the lunar surface, with no reference to the limb.

The lunar photography associated with the new method must have special characteristics. It must be refractor photography, so that there is little possibility of distortion of the images. Furthermore, both the scale and the orientation of the lunar images must be known with a precision similar to that of the heliometer measures. The technique for using star trails to obtain refraction-free photographic coordinates oriented on the moon's hour circle is described in *Comm. LPL* No. 72, while the technique for obtaining and transferring precise focal lengths is given in *Comm.* Nos. 73 and 74.

The following assumes that the photographic coordinates are free of refraction and are oriented with the y-axis along the hour circle through the moon's center of face. Only the geometry of the problem is discussed, since a recent paper by Eckhardt (1965) provides sufficiently precise values for the physical librations ρ, σ, and τ as functions of the ratios β and γ.

2. *Computation of the Selenocentric Coordinates (l'', b'') of the Exposure Station*

The selenocentric coordinates of the exposure station are required for the reduction of each measurement and are usually obtained by differential formulas. These are well suited to desk computations but are not rigorous. In connection with high-speed computers, they have no advantages and should be put aside. We now introduce the usual Eulerian angles, as used by Hayn and Koziel:

θ = inclination of true lunar equator to ecliptic;

ψ = longitude of descending node of true lunar equator on ecliptic;

ϕ = arc measured in the plane of the true lunar equator from its descending node on ecliptic to the moon's first radius.

The lunar first radius corresponds to zero in longitude and latitude in the lunar system of coordinates. The above Eulerian angles are connected to the physical librations ρ, σ, τ by the well-known relations

$$\left.\begin{aligned} \theta &= I + \rho \\ \psi &= \Omega + \tau \\ \phi + \psi &= ☾ + 180° + \tau \end{aligned}\right\}, \qquad (1)$$

in which ☾ is the moon's mean longitude, I is the mean value of θ and is the constant inclination of the Cassini laws, and Ω is the longitude of the ascending node of the lunar orbit. The arc ϕ is most conveniently found from

$$\phi = 180° + (☾ - \Omega) + (\tau - \sigma). \qquad (2)$$

Thus θ, ϕ, and ψ are readily found from ρ, σ, and τ, which in turn are interpolated from Eckhardt's tables.

Now let (l'', b'') be the selenographic longitude and latitude of the exposure station, referred to the moon's true equator and true first radius. Then we have

$$\left.\begin{aligned} \cos(\phi + l'')\cos b'' &= -\cos(\lambda' - \psi - N)\cos\beta' \\ \sin(\phi + l'')\cos b'' &= \sin\beta'\sin\theta - \\ &\qquad \sin(\lambda' - \psi - N)\cos\beta'\cos\theta \\ \sin b'' &= -\sin\beta'\cos\theta - \\ &\qquad \sin(\lambda' - \psi - N)\cos\beta'\sin\theta \end{aligned}\right\}. \qquad (3)$$

In these (λ', β') are the apparent longitude and latitude of the moon at the exposure station while N is the nutation in longitude. The above are nothing more than generalizations of the well-known formulas for the geocentric optical librations, to which they degenerate when $\rho = \sigma = \tau = 0$,

or $\qquad \theta = I,\ \psi = \Omega,\ \phi = 180° + ☾ - \Omega.$

It should be noted that (3) are completely rigorous and introduce no errors beyond those imposed by the limitations of the computer.

3. *Computation of the True Position Angle*

To determine the position angle of the moon's true axis, that is, the perpendicular to the moon's true equator, we must introduce the earth's true equator. Let

i' = inclination of moon's true equator to earth's true equator;

Δ' = arc of true equator of moon from its ascending node on the true equator of the earth to its ascending node on the ecliptic of date;

Ω'' = arc of true equator of the earth from the true equinox of date to the ascending node of the true lunar equator;

ϵ = true obliquity.

From the spherical triangle of Figure 1,

$$\left.\begin{aligned}\sin \triangle' \sin i' &= -\sin \epsilon \sin (\psi + N)\\ \cos \triangle' \sin i' &= \sin \theta \cos \epsilon - \\ &\quad \cos \theta \sin \epsilon \cos (\psi + N)\\ \cos i' &= \cos \theta \cos \epsilon + \\ &\quad \sin \theta \sin \epsilon \cos (\psi + N)\end{aligned}\right\}, \qquad (4)$$

and

$$\left.\begin{aligned}\sin \Omega'' \sin i' &= -\sin \theta \sin (\psi + N)\\ \cos \Omega'' \sin i' &= \cos \theta \sin \epsilon - \\ &\quad \sin \theta \cos \epsilon \cos (\psi + N)\end{aligned}\right\}. \qquad (5)$$

The quantities i' and $\triangle'$ are rigorously computed from (4), and Ω'' from (5). Once again it may be noted that (4) and (5) are mere generalizations of well-known formulas for the elements of the *mean* lunar equator to which formulas they reduce when $\rho = \sigma = \tau = 0$.

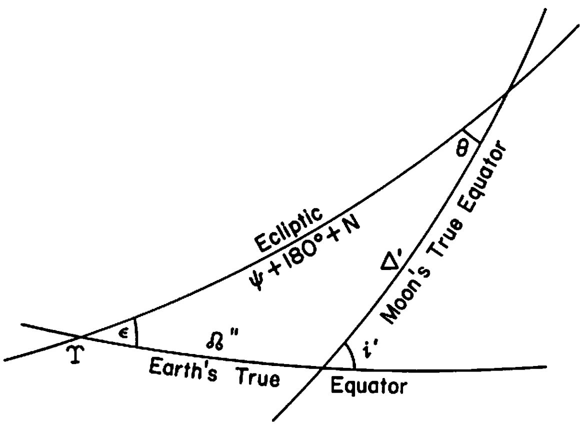

Fig. 1 The moon's true equator.

The position angle C'' of the moon's true axis is found from the generalized expressions for the position angle of the mean axis. These latter are

$$\begin{aligned}\sin C' &= \sin i \cos (l' + \triangle + ☾ - \Omega) \sec \delta'\\ &= -\sin i \cos (\alpha' - \Omega') \sec b',\end{aligned}$$

in which i, $\triangle$, Ω' are the elements of the mean lunar equator, while (l', b') are the topocentric coordinates of the exposure station referred to the mean lunar equator and mean first radius. The obvious generalizations are

$$\left.\begin{aligned}\sin C'' &= -\sin i' \cos (l'' + \triangle' + \phi) \sec \delta'\\ &= -\sin i' \cos (\alpha' - \Omega'') \sec b'\end{aligned}\right\}, \qquad (6)$$

where (α', δ') is the apparent place of the moon.

4. The Preparation of the Observation Equations

Our observation equations differ completely from those of the Bessel-Wichmann scheme since they do not involve the lunar limb. Instead, they are concerned with the coordinate differences between the fundamental point Mösting A and the other measured points, both in the selenodetic system and on the photographs. The point Mösting A has been chosen as the fundamental point merely because good values are known for its coordinates in the usual selenodetic system, but its use is not essential. In our scheme, in its role as a fundamental position, it is replaced by a number of points, all of which have the same importance. These will be termed primary points. The measures are assumed to provide refraction-free rectangular coordinates with the y-axis oriented along the hour circle through the center of the disk.

Now let (U, V, W) be the rectangular selenodetic coordinates *assigned* to the crater Mösting A. These values may be taken from Schrutka-Rechtenstamm (1955) or Koziel (1963). The choice is not important for the purposes of the solution. Let (u, v, w) be the coordinate displacements in the same system from Mösting A to another primary point. Then (u, v, w) are the selenodetic coordinates with Mösting A as origin, and these will be determined by the solution. The unknowns of the solution are thus the moon's elements of rotation, β and γ, and the (u_i, v_i, w_i) of the primary points other than Mösting A.

The coordinates of the primary points in the usual selenodetic coordinate system are (E, F, G) where

$$\left.\begin{aligned}E &= U + u\\ F &= V + v\\ G &= W + w\end{aligned}\right\}. \qquad (7)$$

Let L be the known focal length of the telescope and s' the augmented lunar semidiameter. Let $a, b, \ldots, k$ be the coefficients representing the librations (l'', b'') computed in the usual way, i.e.,

$$a = \cos b'' \sin l'', b = 0, \ldots, k = \cos b'' \cos l''.$$

Then the instantaneous rectangular coordinates of the primary points are (X, Y, Z) where

$$\begin{pmatrix}X\\Y\\Z\end{pmatrix} = \begin{pmatrix}a, & b, & c\\ e, & f, & g\\ i, & j, & k\end{pmatrix} \cdot \begin{pmatrix}E\\F\\G\end{pmatrix}. \qquad (8)$$

The refraction-free photographic coordinates with the y'-axis along the projection of the moon's true axis are (x', y'), where

$$\left.\begin{aligned} x' &= LX \sin s'/(1 - Z \sin s') \\ y' &= LY \sin s'/(1 - Z \sin s') \end{aligned}\right\}. \quad (9)$$

The photographic coordinates with the same origin but with the y''-axis along the hour circle through the center of face are (x'', y''), where

$$\left.\begin{aligned} x'' &= x' \cos C'' - y' \sin C'' \\ y'' &= y' \cos C'' + x' \sin C'' \end{aligned}\right\}. \quad (10)$$

Lastly, the photographic coordinates with the origin at the image of Mösting A and the y-axis parallel to the hour circle through the center of face are (x, y), where

$$\left.\begin{aligned} x &= x'' - x_m'' \\ y &= y'' - y_m'' \end{aligned}\right\}, \quad (11)$$

in which x_m'' and y_m'' are the values of x'' and y'' computed for Mösting A.

The values of x and y, computed as above with assumed values for β and γ and assumed values (u, v, w) for the primary points, are the theoretical values and will be indicated hereafter by (x_c, y_c). The observed values, which are derived directly from the plate measures by subtracting the values for Mösting A from the others, will be indicated by x_o and y_o. The two sets disagree generally because of errors in the assumed values and measures.

5. *The Observation Equations*

The observation equations merely state that the increments to β and γ and to u, v, and w of each primary point must be such as to cancel the differences between the theoretical and observed values of x and y. Hence, they are

$$\left.\begin{aligned} \frac{\partial x}{\partial \beta}\delta\beta + \frac{\partial x}{\partial \gamma}\delta\gamma + \frac{\partial x}{\partial u}\delta u + \frac{\partial x}{\partial v}\delta v + \frac{\partial x}{\partial w}\delta w \\ = x_o - x_c \\ \frac{\partial y}{\partial \beta}\delta\beta + \frac{\partial y}{\partial \gamma}\delta\gamma + \frac{\partial y}{\partial u}\delta u + \frac{\partial y}{\partial v}\delta v + \frac{\partial y}{\partial w}\delta w \\ = y_o - y_c \end{aligned}\right\}. \quad (12)$$

There is one such pair of equations for each measured point on each plate, not counting Mösting A.

The calculation of the partial derivatives in (12) in analytical form is a rather slippery matter, as a glance through the parallel sections in Hayn (1904) reveals. Furthermore, the complication of evaluating the validity of the necessary approximations arises. Fortunately, the characteristics of the high-speed computer make these formal differentiations and subsequent approximations quite unnecessary.

Let $\Delta\beta = \Delta\gamma = 0.0001$ and $\Delta u = \Delta v = \Delta w = 0.001$. Instead of computing the single point (u, v, w) with the single pair (β, γ), we program the computations to compute (x, y) for six points.

Let

(x_c, y_c) correspond to (β, γ, u, v, w),
(x_β, y_β) " " $(\beta + \Delta\beta, \gamma, u, v, w)$,
(x_γ, y_γ) " " $(\beta, \gamma + \Delta\gamma, u, v, w)$,
(x_u, y_u) " " $(\beta, \gamma, u + \Delta u, v, w)$,
(x_v, y_v) " " $(\beta, \gamma, u, v + \Delta v, w)$,
(x_w, y_w) " " $(\beta, \gamma, u, v, w + \Delta w)$.

The derivatives in numerical form are:

$$\frac{\partial x}{\partial \beta} = (x_\beta - x_c)/\Delta\beta, \quad \frac{\partial y}{\partial \beta} = (y_\beta - y_c)/\Delta\beta$$

$$\frac{\partial x}{\partial \gamma} = (x_\gamma - x_c)/\Delta\gamma, \quad \frac{\partial y}{\partial \gamma} = (y_\gamma - y_c)/\Delta\gamma$$

....

$$\frac{\partial x}{\partial w} = (x_w - x_c)/\Delta w, \quad \frac{\partial y}{\partial w} = (y_w - y_c)/\Delta w.$$

The computation is quite direct and unsuited to desk calculations, but since it involves mere repetition of the same routines (1), (2), (3), . . . , (11), it is well adapted to the high-speed computer.

The characteristics of the reduction scheme should now be apparent. Starting from initial guesses for β, γ and u_i, v_i, w_i, the values of l'', b'', C'' are computed for each photograph, then the values of x_c and y_c for each point on each photograph, and finally the values of the partial derivatives in the observation equations. The normal equations are formed and solved for the corrections $\delta\beta$, $\delta\gamma$, δu_i, δv_i, δw_i. The initial values are corrected and a fresh iteration is commenced with the improved guesses. Since β and γ are closely known and since the u_i, v_i, w_i can also be closely estimated, the computation should require few iterations. The flow diagram of Figure 2 illustrates the sequence of the computations.

6. *The Formation of the Normal Equations*

The measures are made in two orientations for each plate, and clearly there are insufficient data to assign different precisions to the measures on different points. It follows that initially, at least, we must assume a uniform variance for all measures on the same plate. The situation is much the same for differences in precision between different plates. Whereas

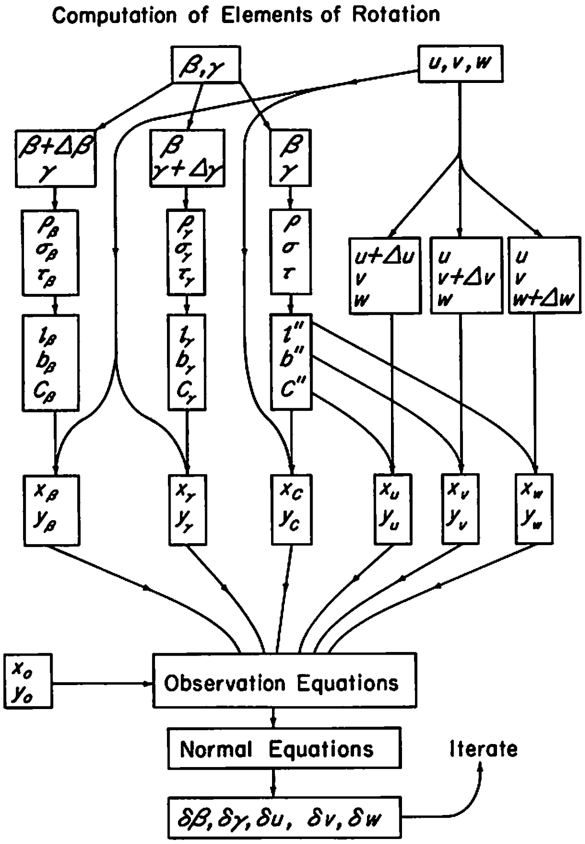

Fig. 2

it may appear to the observer that one plate is better than another, experience shows that the real precision of the measures is by no means correlated with the resolution. There is, however, the possibility of assessing the weight of a star-trailed plate from the residuals of the measures on the trail itself, since the fluctuations of the trail certainly give the order of the seeing displacements. Let us assume for now that weights $p_1, p_2, \ldots, p_m$ are assigned to the m plates. These are computed from

$$p_i = \sigma^2/\sigma_i^2, \tag{13}$$

where σ^2 is the variance of the trail on one plate taken as standard, and σ_i^2 is the variance of the trail on the plate i.

The reduction of the measures on one plate to the image of Mösting A as origin introduces algebraic correlation that cannot be ignored in the formation of the normal equations. Since

$$x = x'' - x_m'',$$
$$y = y'' - y_m'',$$

then if σ_i^2 is the variance for plate i, the variance of each x and y is $2\sigma_i^2$. The covariance between two x values, or between two y values, is σ_i^2.

Assuming now that the x and y errors are statistically independent and that the observation equations are stated in the sequence

$$x_1, y_1, x_2, y_2, x_3, y_3, \ldots,$$

then the covariance matrix C_i between the measures on plate i is

$$C_i = \sigma_i^2 Q_i \tag{14}$$

where

$$\left.\begin{aligned} q_{jj} &= 2, \\ q_{jk} &= 1 \text{ when } (j+k) \text{ is even} \\ q_{jk} &= 0 \text{ when } (j+k) \text{ is odd} \end{aligned}\right\}. \tag{15}$$

The complete covariance matrix for all plates, assuming independence between plates, is

$$\left.\begin{aligned} C &= \operatorname{diag}(C_1, C_2, \ldots, C_m) \\ &= \operatorname{diag}(\sigma_1^2 Q_1, \sigma_2^2 Q_2, \ldots, \sigma_m^2 Q_m) \end{aligned}\right\}. \tag{16}$$

Normalizing this with σ^2, the variance of an observation of unit weight, we obtain the correlation matrix

$$Q = \operatorname{diag}\left(\frac{Q_1}{p_1}, \frac{Q_2}{p_2}, \ldots, \frac{Q_m}{p_m}\right). \tag{17}$$

The reciprocal of this is the generalized weight matrix G, that is,

$$G = \operatorname{diag}(p_1 Q_1^{-1}, p_2 Q_2^{-1}, \ldots, p_m Q_m^{-1}). \tag{18}$$

Hence, the computation of the generalized weight matrix G reduces to the separate inversion of each of the matrices Q_i. These are quite large, their order being $2n$ where n is the number of observed points on the plate, not counting Mösting A. Fortunately, the straight numerical inversion of the Q_i is not necessary. Their structure as defined in (15) is very simple, and it is easily shown that their reciprocals have the same structure. From this consideration, it follows that if R_i is the reciprocal of Q_i, its elements are given by

$$\left.\begin{aligned} r_{jj} &= n/(n+1) \\ r_{jk} &= -\,1/(n+1) \text{ if } (j+k) \text{ is even} \\ r_{jk} &= 0 \qquad\qquad\;\; \text{if } (j+k) \text{ is odd} \end{aligned}\right\}, \tag{19}$$

where again n is the number of measured points not counting Mösting A.

The normal equations can now be formed. Let D be the matrix of derivatives in the observation equations (12), and let Δ be the column matrix of unknowns

$$\delta\beta, \delta\gamma, \delta u_1, \delta v_1, \delta w_1, \delta u_2, \delta v_2, \delta w_2, \ldots.$$

Then the observation equations can be written in matrix form as

$$D\Delta = X \tag{20}$$

where X is the column matrix of the differences on the right-hand sides of (12). The normal equations, taking correlation into account, are

$$(D^T G D)\Delta = D^T G X, \tag{21}$$

and the variance of an observation of unit weight is estimated from

$$\sigma^2 = \frac{v^T G v}{2\Sigma n - 3k - 2}, \tag{22}$$

in which k is the number of points whose coordinates are determined, n is the number of points observed on each plate not counting Mösting A, and v is the column matrix of residuals. Assuming that β and γ are taken as the first unknowns, the diagonal elements of the inverse normal matrix

$$(D^T G D)^{-1}$$

provide the variances in units of σ^2 of β, γ, and then of the coordinates u_i, v_i, w_i in sequence.

7. *Concluding Remarks*

The method sketched above, which is being used for the measures on Yerkes star-trailed plates, appears to represent the first attempt to break away from the limitations of classical selenodesy, namely the limitations of the heliometer and the Bessel-Wichmann scheme, and the limitations of the computing methods which appear to result from using desk computers. It is true that Khabibullin (1958) at Kazan used measures on photographs to determine the constants of the moon's rotation, but he used Mösting A and the limb in the traditional way. It is also true that Koziel (1963) used a large computer for reducing four heliometer series, but again the computations were differential and not specially adapted to the high-speed computer.

The reductions take no account of free librations because of the relative shortness of the period of photography. The amplitude and phase of the free libration in longitude found by Koziel could be used, but it is much too early to attempt this kind of refinement in the reductions of material whose precision is yet unknown.

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