Secular resonance sweeping of the main asteroid belt during planet migration

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ABSTRACT

We calculate the eccentricity excitation of asteroids produced by the sweeping $\nu_6$ secular resonance during the epoch of planetesimal-driven giant planet migration in the early history of the solar system. We derive analytical expressions for the magnitude of the eccentricity change and its dependence on the sweep rate and on planetary parameters; the $\nu_6$ sweeping leads to either an increase or a decrease of eccentricity depending on an asteroid’s initial orbit. Based on the slowest rate of $\nu_6$ sweeping that allows a remnant asteroid belt to survive, we derive a lower limit on Saturn’s migration speed of $\sim 0.15$ AU My$^{-1}$ during the era that the $\nu_6$ resonance swept through the inner asteroid belt; this rate limit scales with the square of Saturn’s eccentricity, and could be lower if Saturn’s eccentricity were lower at early times. Applied to an ensemble of asteroids, our calculations show that a prior single-peaked distribution of asteroid eccentricities would be transformed into a double-peaked eccentricity distribution. Examination of the orbital data of main belt asteroids reveals that the proper eccentricities of the known large ($H \leq 10.8$) asteroids do indeed have a
double-peaked distribution. Our theoretical analysis yields two possible dynamical states of the pre-migration asteroid belt: a dynamically cold state (mean eccentricities $\sim 0.05$) and a dynamically hot state (mean eccentricities $\sim 0.3$), each of which has specific implications for the early history of the asteroid belt and for the bombardment history of the inner solar system.

Subject headings: celestial mechanics — minor planets, asteroids: general — planets and satellites: dynamical evolution and stability — solar system: general

1. Introduction

Evidence in the structure of the Kuiper Belt suggests that the outer solar system experienced a phase of planetesimal-driven migration in its early history (Fernandez & Ip 1984; Malhotra 1993, 1995; Hahn & Malhotra 1999; Levison et al. 2008). Planetesimal-driven giant planet migration has been suggested as a cause of the Late Heavy Bombardment (LHB) (Gomes et al. 2005; Strom et al. 2005). Such migration would have enhanced the impact flux onto the terrestrial planets in two ways. First, many of the icy planetesimals scattered by the giant planets would have crossed the orbits of the terrestrial planets. Second, as the giant planets migrated, locations of mean motion and secular resonances would have swept across the asteroid belt, raising the eccentricities of asteroids to planet-crossing values.

Recently, Minton & Malhotra (2009) showed that the patterns of depletion observed in the asteroid belt are consistent with the effects of sweeping of resonances during the migration of the outer giant planets. In this paper we explore one important effect that planet migration would have had on the asteroid belt: asteroid eccentricity excitation by the sweeping of the $\nu_6$ secular resonance.

Sweeping, or scanning, secular resonances have been analyzed in a number of previous works. Sweeping secular resonances due to the changing quadrupole moment of the Sun during solar spin-down have been explored as a possible mechanism for explaining the eccentricity and inclination of Mercury (Ward et al. 1976). Secular resonance sweeping due to the effects of the dissipating solar nebula just after planet formation has been investigated as a possible mechanism for exciting the orbital eccentricities of Mars and of the asteroid belt (Heppenheimer 1980; Ward 1981). The dissipating massive gaseous solar nebula would have altered the secular frequencies of the solar system planets in a time-dependent way, causing locations of secular resonances to possibly sweep across the inner solar system, thereby exciting asteroids into the eccentric and inclined orbits that are observed today. This
mechanism was revisited by Nagasawa et al. (2000), who incorporated a more sophisticated treatment of the nebular dispersal. However, O’Brien et al. (2007) have argued that the excitation (and clearing) of the primordial asteroid belt involving secular resonance sweeping due to the dispersion of the solar nebula may not be the best explanation for the asteroid belt.

Gomes (1997) has investigated the effect of the sweeping of the $\nu_6$ and $\nu_{16}$ secular resonances on the asteroid belt due to the migration of the giant planets, but he (in common with other previous investigations of secular resonance sweeping) considered only the special case of initially circular orbits. In the work we present in this paper, we consider that more general case of non-zero initial eccentricities; our analysis yields qualitatively new results and provides new insights into the dynamical history of the asteroid belt. This extends the work of Ward et al. (1976) and Gomes (1997) in developing analytical approximations to the effect that sweeping secular resonances have on asteroid orbits. In doing so, we have develop an explicit relationship between the migration rate of the giant planets, the initial eccentricity of the asteroid and its initial longitude of perihelion, and the final eccentricity of the asteroid. We show that for initial non-zero eccentricity, the sweeping of the $\nu_6$ resonances can either increase or decrease asteroid eccentricities. Examining the orbits of observed main belt asteroids we find evidence for a double-peaked eccentricity distribution; this supports the case for a history of $\nu_6$ sweeping. Quantitative comparison of our analytical theory with the semimajor axis and eccentricity distribution of asteroids yields new constraints on the timescale of planet migration.

We note that although our analysis is carried out in the specific context of the sweeping $\nu_6$ resonance during the phase of planetesimal-driven migration of Jupiter and Saturn, the techniques developed here may be extended to other similar problems, for example, the sweeping of the inclination-node $\nu_{16}$ resonance in the main asteroid belt, the $\nu_8$ secular resonance in the Kuiper belt, and farther afield, the sweeping of secular resonances in circumstellar or galactic disks.

2. Analytical theory of a sweeping secular resonance

We adopt a simplified model in which a test particle (asteroid) is perturbed only by a single resonance, the $\nu_6$ resonance resonance. We use a system of units where the mass is in solar masses, the semimajor axis is in units of AU, and the unit of time is $(1/2\pi)y$. With this choice, the gravitational constant, $G$, is unity. An asteroid’s secular perturbations close to a secular resonance can be described by the following Hamiltonian function (Malhotra 1998):

$$H_{sec} = -g_0J + \epsilon\sqrt{2J}\cos(w_p - \varpi),$$

(1)
where \( w_p = g_p t + \beta_p \) describes the phase of the \( p \)-th eigenmode of the linearized eccentricity-pericenter secular theory for the Solar system planets (Murray & Dermott 1999), \( g_p \) is the associated eigenfrequency, \( \varpi \) is the asteroid’s longitude of perihelion, \( J = \sqrt{a} \left( 1 - \sqrt{1 - e^2} \right) \) is the canonical generalized momentum which is related to the asteroid’s orbital semimajor axis \( a \) and eccentricity \( e \); \( -\varpi \) and \( J \) are the canonically conjugate pair of variables in this 1-degree-of-freedom Hamiltonian system. The coefficients \( g_0 \) and \( \epsilon \) are given by:

\[
\begin{align*}
g_0 &= \frac{1}{4a^{3/2}} \sum_j \alpha_j^2 b^{(1)}_{3/2}(\alpha_j) m_j, \\
\epsilon &= \frac{1}{4a^{5/4}} \sum_j \alpha_j^2 b^{(2)}_{3/2}(\alpha_j) m_j E_j^{(p)},
\end{align*}
\]

where the subscript \( j \) refers to a planet, \( E_j^{(p)} \) is the amplitude of the \( g_p \) mode in the \( j \)-th planet’s orbit, \( \alpha_j = \min \{ a/a_j, a_j/a \} \), \( m_j \) is the ratio of the mass of planet \( j \) to the Sun, and \( b^{(1)}_{3/2}(\alpha_j) \) and \( b^{(2)}_{3/2}(\alpha_j) \) are Laplace coefficients; the sum is over all major planets. The summations in equations (2)–(3) are over the 8 major planets, for the greatest accuracy; however, we will adopt the simpler two-planet model of the Sun-Jupiter-Saturn in §3, in which case we sum over only the indices referring to Jupiter and Saturn.

With fixed values of the planetary parameters, \( g_0, g_p \) and \( \epsilon \) are constant parameters in the Hamiltonian given in equation (1). However, during the epoch of giant planet migration, the planets’ semimajor axes change secularly with time, so that \( g_0, g_p \) and \( \epsilon \) become time-dependent parameters. In the analysis below, we neglect the time-dependence of \( g_0 \) and \( \epsilon \), and adopt a simple prescription for the time-dependence of \( g_p \) (see equation 8 below). This approximation is physically motivated: the fractional variation of \( g_0 \) and \( \epsilon \) for an individual asteroid is small compared to the effects of the “small divisor” \( g_0 - g_p \) during the \( \nu_6 \) resonance sweeping event.

It is useful to make a canonical transformation to new variables \((\phi, P)\) defined by the following generating function,

\[
\mathcal{F}(-\varpi, P, t) = (w_p(t) - \varpi)P
\]

Thus, \( \phi = \partial \mathcal{F}/\partial P = (w_p(t) - \varpi) \) and \( J = -\partial \mathcal{F}/\partial \varpi = P \). The new Hamiltonian function is \( \tilde{H}_{sec} = H_{sec} + \partial \mathcal{F}/\partial t \),

\[
\tilde{H}_{sec} = (\dot{w}_p(t) - g_0)J + \epsilon \sqrt{2J} \cos \phi,
\]

where we have retained \( J \) to denote the canonical momentum, since \( P = J \). It is useful to make a second canonical transformation to canonical eccentric variables,

\[
x = \sqrt{2J} \cos \phi, \quad y = -\sqrt{2J} \sin \phi,
\]

\[\text{(6)}\]
where \( x \) is the canonical coordinate and \( y \) is the canonically conjugate momentum. The Hamiltonian expressed in these variables is

\[
\tilde{H}_{sec} = (\dot{w}_p(t) - g_0) \frac{x^2 + y^2}{2} + \epsilon x.
\] (7)

As discussed above, during planetary migration, the secular frequency \( g_p \) is a slowly varying function of time. We approximate its rate of change, \( \dot{g}_p = 2\lambda \), as a constant, so that

\[
\dot{w}_p(t) = g_{p,0} + 2\lambda t.
\] (8)

We define \( t = 0 \) as the epoch of exact resonance crossing, so that \( g_{p,0} = g_0 \) (cf. Ward et al. 1976). Then, \( \dot{w}_p(t) - g_0 = 2\lambda t \), and the equations of motion from the Hamiltonian of equation (7) can be written as:

\[
\begin{align*}
\dot{x} &= 2\lambda ty, \\
\dot{y} &= -2\lambda tx - \epsilon.
\end{align*}
\] (9-10)

These equations of motion form a system of linear, nonhomogenous differential equations, whose solution is a linear combination of a homogeneous and a particular solution. The homogeneous solution can be found by inspection, giving:

\[
\begin{align*}
x_h(t) &= c_1 \cos \lambda t^2 + c_2 \sin \lambda t^2, \\
y_h(t) &= -c_1 \sin \lambda t^2 + c_2 \cos \lambda t^2,
\end{align*}
\] (11-12)

where \( c_1 \) and \( c_2 \) are constant coefficients. We use the method of variation of parameters to find the particular solution. Accordingly, we replace the constants \( c_1 \) and \( c_2 \) in the homogeneous solution with functions \( A(t) \) and \( B(t) \), to seek the particular solution of the form

\[
\begin{align*}
x_p(t) &= A(t) \cos \lambda t^2 + B(t) \sin \lambda t^2, \\
y_p(t) &= -A(t) \sin \lambda t^2 + B(t) \cos \lambda t^2.
\end{align*}
\] (13-14)

Substituting this into the equations of motion we now have:

\[
\begin{align*}
\dot{A} \cos \lambda t^2 + \dot{B} \sin \lambda t^2 &= 0, \\
-\dot{A} \sin \lambda t^2 + \dot{B} \cos \lambda t^2 &= -\epsilon;
\end{align*}
\] (15-16)

therefore

\[
\begin{align*}
\dot{A} &= \epsilon \sin \lambda t^2, \\
\dot{B} &= -\epsilon \cos \lambda t^2.
\end{align*}
\] (17-18)
Equations (17) and (18) cannot be integrated analytically, but they can be expressed in terms of Fresnel integrals (Zwillinger 1996). The Fresnel integrals are defined as follows:

\[ S(t) = \int_0^t \sin t'^2 dt' , \quad (19) \]
\[ C(t) = \int_0^t \cos t'^2 dt' . \quad (20) \]

and have the following properties:

\[ S(-t) = -S(t) , \quad (21) \]
\[ C(-t) = -C(t) , \quad (22) \]
\[ S(\infty) = C(\infty) = \sqrt{\frac{\pi}{8}} . \quad (23) \]

Therefore

\[ A(t) = \frac{\epsilon}{\sqrt{|\lambda|}} S \left( t\sqrt{|\lambda|} \right) , \quad (24) \]
\[ B(t) = -\frac{\epsilon}{\sqrt{|\lambda|}} C \left( t\sqrt{|\lambda|} \right) . \quad (25) \]

We denote initial conditions with a subscript \( i \), and write the solution to equations (9) and (10) as

\[
x(t) = x_i \cos \left[ \lambda \left( t^2 - t_i^2 \right) \right] + y_i \sin \left[ \lambda \left( t^2 - t_i^2 \right) \right] \\
+ \frac{\epsilon}{\sqrt{|\lambda|}} \left[ (S - S_i) \cos \lambda t^2 - (C - C_i) \sin \lambda t^2 \right] , \quad (26) \\
y(t) = -x_i \sin \left[ \lambda \left( t^2 - t_i^2 \right) \right] + y_i \cos \left[ \lambda \left( t^2 - t_i^2 \right) \right] \\
- \frac{\epsilon}{\sqrt{|\lambda|}} \left[ (C - C_i) \cos \lambda t^2 + (S - S_i) \sin \lambda t^2 \right] . \quad (27) 
\]

Because the asteroid is swept over by the secular resonance at time \( t = 0 \), we can calculate the changes in \( x,y \) by letting \( t_i = -t_f \) and evaluating the coefficients \( C_i, C_f, S_i, S_f \) far from resonance passage, i.e., for \( t_f \sqrt{|\lambda|} \gg 1 \), by use of equation (23). Thus we find

\[
x_f = x_i + \epsilon \sqrt{\frac{\pi}{2|\lambda|}} \left[ \cos \lambda t_i^2 - \sin \lambda t_i^2 \right] , \quad (28) \\
y_f = y_i - \epsilon \sqrt{\frac{\pi}{2|\lambda|}} \left[ \cos \lambda t_i^2 + \sin \lambda t_i^2 \right] . \quad (29) 
\]
The new value of $J$ after resonance passage is therefore given by

$$J_f = \frac{1}{2} (x_f^2 + y_f^2)$$

$$= \frac{1}{2} (x_i^2 + y_i^2) + \frac{\pi \epsilon^2}{2|\lambda|} + \epsilon \sqrt{\pi} \left[ x_i (\cos \lambda t_i^2 \sin \lambda t_i^2) - y_i (\cos \lambda t_i^2 \sin \lambda t_i^2) \right]$$

$$= J_i + \frac{\pi \epsilon^2}{2|\lambda|} + \epsilon \sqrt{\frac{2\pi J_i}{|\lambda|}} \cos(\phi_i - \lambda t_i^2 - \frac{\pi}{4}). \quad (30)$$

With a judicious choice of the initial time, $t_i$, and without loss of generality, the cosine in the last term becomes $\cos \omega_i$, and therefore

$$J_f = J_i + \frac{\pi \epsilon^2}{2|\lambda|} + \epsilon \sqrt{\frac{2\pi J_i}{|\lambda|}} \cos \omega_i. \quad (31)$$

The asteroid’s semimajor axis $a$ is unchanged by the secular perturbations; thus, the changes in $J$ reflect changes in the asteroid’s eccentricity $e$. For asteroids with non-zero initial eccentricity, the phase dependence in equation (31) means that secular resonance sweeping can potentially both excite and damp orbital eccentricities. We also note that the magnitude of eccentricity change is inversely related to the speed of planet migration.

For small $e$, we can use the approximation $J \simeq \frac{1}{2} \sqrt{ae^2}$. Considering all possible values of $\cos \omega_i \in \{-1, +1\}$, an asteroid with initial eccentricity $e_i$ that is swept by the $\nu_6$ resonance will have a final eccentricity in the range $e_{\min}$ to $e_{\max}$, where

$$e_{\min,\max} \simeq |e_i \pm \delta_e|, \quad (32)$$

and

$$\delta_e \equiv \epsilon \sqrt{\frac{\pi}{|\lambda| \sqrt{a}}} \quad (33)$$

It is useful to examine the implications of equations (31)–(33): (i) Initially circular orbits become eccentric, with a final eccentricity $\delta_e$. (ii) An ensemble of orbits with the same $a$ and initial $e$ but random orientations of pericenter are transformed into an ensemble that has a distribution of eccentricities in the range $e_{\min}$ to $e_{\max}$; this range is not uniformly distributed because of the $\cos \omega_i$ dependence in equation (31), rather the distribution peaks at the extreme values (see Fig. 4 below). (iii) We can infer from the latter effect that an ensemble of asteroids having an initial distribution of eccentricities which is approximately a single-peaked Gaussian and random orientations of pericenter would be transformed into one with a double-peaked eccentricity distribution.
3. $\nu_6$ sweeping of the Main Asteroid Belt

3.1. Parameters

In order to apply the above analysis to the problem of the $\nu_6$ resonance sweeping through the asteroid belt, we must obtain values for the parameter $\epsilon$ (equation 3), for asteroids with semimajor axis values in the main asteroid belt. We must also find the location of the $\nu_6$ resonance as a function of the semimajor axes of the giant planets orbits. The location of the $\nu_6$ resonance is defined as the semimajor axis, $a_{\nu_6}$, where the rate, $g_0$ (equation 2), of pericenter precession of a massless particle (or asteroid) is equal to the $g_6$ eigenfrequency of the solar system. In the current solar system, the $g_6$ frequency is associated with the secular mode with the most power in Saturn’s eccentricity-pericenter variations. During the epoch of planetesimal-driven planet migration, Jupiter migrated by only a small amount but Saturn likely migrated significantly more (Fernandez & Ip 1984; Malhotra 1995; Tsiganis et al. 2005), so we expect that the variation in location of the $\nu_6$ secular resonance is most sensitive to Saturn’s semimajor axis. We therefore adopt a simple model of planet migration in which Jupiter is fixed at 5.2 AU and only Saturn migrates. We neglect the effects of the ice giants Uranus and Neptune, as well as secular effects due to the more massive trans-Neptunian planetesimal disk and the more massive pre-migration asteroid belt. In this simplified model, the $g_6$ frequency varies with time as Saturn migrates, so $g_6$ is solely a function of Saturn’s semimajor axis. In contrast with the variation of $g_6$, there is negligible variation of the asteroid’s precession rate, $g_0$, as Saturn migrates.

For fixed planetary semimajor axes, the Laplace-Lagrange secular theory provides the secular frequencies and orbital element variations of the planets. This is a linear perturbation theory, in which the disturbing function is truncated to secular terms of second order in eccentricity and first order in mass (Murray & Dermott 1999). In the planar two-planet case, the secular perturbations of planet $j$, where $j = 5$ is Jupiter and $j = 6$ is Saturn, are described by the following disturbing function:

$$R_j = \frac{n_j}{a_j^2} \left[ \frac{1}{2} A_{jj} e_j^2 + A_{jk} e_5 e_6 \cos(\varpi_5 - \varpi_6) \right],$$  \hspace{1cm} (34)

where $n$ is the mean motion, and $A$ is a matrix with elements

$$A_{jj} = +n_j \frac{1}{3} \frac{m_k}{M_0 + m_j} \alpha_{jk} \bar{\alpha}_{jk} b_{3/2}^{(1)}(\alpha_{jk}),$$

$$A_{jk} = -n_j \frac{1}{3} \frac{m_k}{M_0 + m_j} \alpha_{jk} \bar{\alpha}_{jk} b_{3/2}^{(2)}(\alpha_{jk}),$$

for $j = 5, 6, k = 6, 5$, and $j \neq k$; $\alpha_{jk} = \min\{a_j/a_k, a_j/a_k\}$, and

$$\bar{\alpha}_{jk} = \begin{cases} 1 & : a_j > a_k \\ a_j/a_k & : a_j < a_k. \end{cases}$$  \hspace{1cm} (37)
The secular motion of the planets is then described by a set of linear differential equations for the eccentricity vectors, $\varepsilon_j(\sin \varpi_j, \cos \varpi_j) \equiv (h_j, k_j)$,

$$
\dot{h}_j = + \sum_{p=5}^{6} A_{pj} k_j, \quad \dot{k}_j = - \sum_{p=5}^{6} A_{pj} h_j.
$$

For fixed planetary semimajor axes, the coefficients are constants, and the solution is given by a linear superposition of eigenmodes:

$$
\{h_j, k_j\} = \sum_p E^{(p)}_j \{\cos(g_p t + \beta_p), \sin(g_p t + \beta_p)\},
$$

where $g_p$ are the eigenfrequencies of the matrix $A$ and $E^{(p)}_j$ are the corresponding eigenvectors; the amplitudes of the eigenvectors and the phases $\beta_p$ are determined by initial conditions. In our 2-planet model, the secular frequencies $g_5$ and $g_6$ depend on the relative masses of Jupiter and Saturn to the Sun and on the semimajor axes of Jupiter and Saturn.

For the current semimajor axes of Jupiter and Saturn the Laplace-Lagrange theory gives frequency values $g_5 = 3.7'' y^{-1}$ and $g_6 = 22.3'' y^{-1}$, which are lower than the more accurate values given by Brouwer & van Woerkom (1950) by 14% and 20%, respectively (Laskar 1988). Brouwer & van Woerkom (1950) achieved their more accurate solution by incorporating higher order terms in the disturbing function involving $2\lambda_5 - 5\lambda_6$, which arise due to Jupiter and Saturn’s proximity to the 5:2 resonance (the so-called “Great Inequality”). By doing an accurate numerical analysis (described below), we found that the effect of the 5:2 resonance is only important over a very narrow range in Saturn’s semimajor axis. More significant is the perturbation owing to the 2:1 near-resonance of Jupiter and Saturn. Malhotra et al. (1989) developed corrections to the Laplace-Lagrange theory to account for the perturbations from $n + 1 : n$ resonances in the context of the Uranian satellite system. Applying that approach to our problem, we find that the 2:1 near-resonance between Jupiter and Saturn leads to zeroth order corrections to the elements of the $A$ matrix. Including these corrections, we determined the secular frequencies for a range of values of Saturn’s semimajor axis; the result for $g_6$ is shown in Fig. 1 (dashed line).

We have also calculated values for the eccentricity-pericenter eigenfrequencies by direct numerical integration of the two-planet, planar solar system. In these simulations, Jupiter’s initial semimajor axis was 5.2 AU, Saturn’s semimajor axis, $a_6$, was one of 233 values in the range 7.3–10.45 AU, initial eccentricities of Jupiter and Saturn were 0.05, and initial inclinations were zero. The initial longitude of pericenter and mean anomalies of Jupiter were $\varpi_{5,i} = 15^\circ$ and $\lambda_{5,i} = 92^\circ$, and Saturn were $\varpi_{6,i} = 338^\circ$ and $\lambda_{6,i} = 62.5^\circ$. In each case, the planets orbits were integrated for 100 myr, and a Fourier transform of the time series...
of the \( \{h_j, k_j\} \) yields their spectrum of secular frequencies. For regular (non-chaotic) orbits, the spectral frequencies are well defined and are readily identified with the frequencies of the secular solution. The \( g_6 \) frequency as a function of Saturn’s semimajor axis was obtained by this numerical analysis; the result is shown by the solid line in Fig. 1.

The comparison between the numerical analysis and the analytical solution indicates that the linear secular theory (with corrections for the 2:1 near-resonance) is an adequate model for the variation in \( g_6 \) as a function of \( a_6 \), and we adopted this for the needed computations. The value of \( a_{\nu_6} \) as a function of Saturn’s semimajor axis was thus found by solving for the value of asteroid semimajor axis where \( g_0 = g_6 \); \( g_0 \) was calculated using equation (2) and \( g_6 \) is the eigenfrequency associated with the \( p = 6 \) eigenmode (at each value of Saturn’s semimajor axis). The result is shown in Fig. 1b.

We also used the analytical secular theory to calculate the eigenvector components \( E_j^{(6)} \) in the secular solution of the 2-planet system, for each value of Saturn’s semimajor axis. We adopted the same values for the initial conditions of Jupiter and Saturn as in the direct numerical integrations discussed above. Finally, we computed the values of the parameter \( \epsilon \) at each location \( a_{\nu_6} \) of the secular resonance. The result is plotted in Fig. 2.

### 3.2. Four test cases

We checked the results of our analytical model against four direct numerical simulations of the restricted four-body problem (the Sun-Jupiter-Saturn system with test particle asteroids) in which the test particles in the asteroid belt are subjected to the effects of a migrating Saturn. In each of the four simulations, 30 test particles were placed at 2.3 AU and given different initial longitudes of pericenter spaced 10° apart. Jupiter and Saturn were the only planets included, and the asteroids were approximated as massless test particles. The current Solar System values of the eccentricity of Jupiter and Saturn were adopted and inclinations were set to zero. An external acceleration was applied to Saturn to cause it to migrate outward starting at 8.5 AU at the desired rate. The numerical integration was performed with an implementation of a symplectic mapping (Wisdom & Holman 1991; Saha & Tremaine 1992), and the integration stepsize was 0.01 y. The only parameters varied between each of the four simulations were the initial osculating eccentricities of the test particles, \( e_i \), and the migration speed of Saturn, \( \dot{a}_6 \). The parameters explored were:

- a) \( e_i = 0.2, \dot{a}_6 = 1.0 \text{ AU My}^{-1} \);
- b) \( e_i = 0.2, \dot{a}_6 = 0.5 \text{ AU My}^{-1} \);
c) $e_i = 0.1, \dot{a}_6 = 1.0 \text{ AU My}^{-1}$; 

d) $e_i = 0.3, \dot{a}_6 = 1.0 \text{ AU My}^{-1}$.

Two aspects of the analytical model were checked. First, the perturbative equations of motion, equations (9) and (10), were numerically integrated, and their numerical solution compared with the numerical solution from the direct numerical integration of the full equations of motion. For the perturbative solution, we adopted values for $\lambda$ that were approximately equivalent to the values of $\dot{a}_6$ in the full numerical integrations. Second, the eccentricity bounds predicted by the analytical theory, equation (32), were compared with both numerical solutions. The results of these comparisons for the four test cases are shown in Fig. 3. We find that the analytically predicted values of the maximum and minimum final eccentricities (shown as horizontal dashed lines) are in excellent agreement with the final values of the eccentricities found in the perturbative numerical solution, and in fairly good agreement with those found in the full numerical solution. Not surprisingly, we find that the test particles in the full numerical integrations exhibit somewhat more complicated behavior than the perturbative approximation, and equation (32) somewhat underpredicts the maximum final eccentricity: this may be due to higher order terms in the disturbing function that have been neglected in the perturbative analysis and which become more important at high eccentricity; effects due to close encounters with Jupiter also become important at the high eccentricities.

4. Double-peaked asteroid eccentricity distribution

An important implication of equation (31) is that if the asteroid belt were initially dynamically cold, that is asteroids were on nearly circular orbits prior to secular resonance sweeping, then the asteroids would be nearly uniformly excited to a narrow range of final eccentricities, the value of which would be determined by the rate of resonance sweeping. Because asteroids having eccentricities above planet-crossing values would be unlikely to survive to the present day, it follows that an initially cold asteroid belt which is uniformly excited by the $\nu_6$ sweeping will either lose all its asteroids or none. On the other hand, an initially excited asteroid belt, that is a belt with asteroids that had finite non-zero eccentricities prior to the $\nu_6$ sweeping, would have asteroids’ final eccentricities bounded by equation (32), allowing for partial depletion and also broadening of its eccentricity distribution. Examination of the observed eccentricity distribution of asteroids in light of these considerations leads to some important new insights about the asteroid belt.

To do this, we need to obtain the eccentricity distribution free of observational bias. We
therefore obtained data on the observationally complete sample of asteroids with absolute magnitude cut-off $H \leq 10.8$; we also excluded from this set the members of collisional families as identified by Nesvorný et al. (2006). This sample is a good approximation to a complete set of asteroids that have been least perturbed by either dynamical evolution or collisional evolution since the epoch of the last major dynamical event that occurred in this region of the solar system; therefore this sample likely preserves best the post-migration orbital distribution of the asteroid belt. The proper eccentricity distribution of these asteroids is shown in Fig. 5. The eccentricity distribution of asteroids has usually been described in the literature by simply quoting its mean value (and sometimes a dispersion) (Murray & Dermott 1999; O’Brien et al. 2007), and indeed, a single Gaussian distribution provides a fairly good fit, as indicated in Fig. 5. However, we also note a clear indication of a double-peaked distribution in the observed population. Our best fit single-peaked Gaussian distribution to the $H \leq 10.8$ asteroids has a mean, $\mu_e$ and standard deviation, $\sigma_e$, given by $\mu_e = 0.135 \pm 0.00013$ and $\sigma_e = 0.0716 \pm 0.00022$. Our best fit double-peaked Gaussian distribution (given as two symmetrical Gaussians with the same standard deviation, but with offset mean values) of the same population of asteroids has the following parameters:

\[
\begin{align*}
\mu_{e,1}' &= 0.0846 \pm 0.00011, \\
\mu_{e,2}' &= 0.185 \pm 0.00012, \\
\sigma_e' &= 0.0411 \pm 0.00020.
\end{align*}
\]

More details of how these fits were obtained can be found in Appendix A.

5. A Constraint on Saturn’s migration rate

By relating the $g_6$ secular frequency to the semimajor axis of Saturn, $\dot{a}_6$, we used the results of our analytical model to set limits on the rate of migration of Saturn, $\dot{a}_6$. We used the results of our analytical model to set limits on the rate of migration of Saturn, with the caveat that many important effects are ignored, such as asteroid-Jupiter mean motion resonances, and Jupiter-Saturn mean motion resonances with the exception of the 2:1 resonance. We have confined our analysis to only the inner main belt, between 2.2–2.8 AU. Beyond 2.8 AU strong jovian mean motion resonance become more numerous. The migration of Jupiter would have caused strong jovian mean motion resonances to sweep the asteroid belt, causing additional depletion beyond that of the sweeping $\nu_6$ resonance. There is evidence from the distribution of asteroids in the main belt that the sweeping of the 5:2, 7:3, and 2:1 jovian mean motion resonances may have depleted the main belt (Minton & Malhotra 2009). A further complication is that sweeping
jovian mean motion resonances may have also trapped icy planetesimals that entered the asteroid belt region from their source region beyond Neptune (Levison et al. 2009). The effects of these complications are reduced when we only consider the inner asteroid belt. From Fig. 1b, the $\nu_6$ would have swept the inner asteroid belt region between 2.2–2.8 AU when Saturn was between $\sim$ 8.5–9.2 AU. Therefore the limits on $\dot{a}_6$ that we set using the inner asteroid belt as a constraint are only applicable for this particular portion of Saturn’s migration history.

An estimated final eccentricity as a function of initial asteroid semimajor axis, initial asteroid eccentricity, and the migration rate of Saturn is shown in Fig. 7. The larger the initial asteroid eccentricities, the wider the bounds in their final eccentricities. If we adopt the criterion that an asteroid is lost from the main belt when it achieves a planet-crossing orbit (that is, crossing the orbits of either Jupiter or Mars) and that initial asteroid eccentricities were therefore confined to $\lesssim 0.4$, then from Fig. 7 Saturn’s migration rate while the $\nu_6$ resonance was passing through the inner asteroid belt must have been $\dot{a}_6 \gtrsim 0.15$ AU My$^{-1}$. This model suggests that if Saturn’s migration rate had been slower than 0.15 AU My$^{-1}$ while it was migrating across $\sim 8.5$–9.2 AU, then the inner asteroid belt would have been completely swept clear of asteroids by the $\nu_6$ resonance.

In light of our analysis and the observed dispersion of eccentricities in the asteroid belt (Fig. 5), we can immediately conclude that the pre-migration asteroid belt had a finite initial eccentricity distribution. This is perhaps a trivial inference, but it is consistent with planetesimal accretion and asteroid and planet formation theory which indicates that the asteroids were modestly excited at the end of their formation (e.g., O’Brien et al. 2007).

The observed double-peaked eccentricity distribution of the main asteroid belt may be used to further constrain the migration rate of Saturn. If the pre-sweeping asteroid belt had a Gaussian eccentricity distribution, then the lower peak of the post-sweeping asteroid belt should be equal to the lower bound of equation (32). Assuming that the pre-migration asteroid belt had a single-peaked eccentricity distribution, we used the analytical theory to make a rough estimate of the parameter $\lambda$ (and hence $\dot{a}_6$) that would yield a final distribution with lower peak near 0.09 and upper peak near 0.19 (which is similar to the best-fit double Gaussian in Fig. 5). Applying equation (32), we see that there are two possible solutions: $\langle e_i \rangle = 0.14, \delta_e = 0.05$ and $\langle e_i \rangle = 0.05, \delta_e = 0.14$. A corresponding migration rate of Saturn can be estimated from the value of $\delta_e$ using equation (33), and the relationships plotted in Figs. 1 and 2. The former solution ($\delta_e = 0.05$) requires a migration rate for Saturn of $\dot{a}_6 = 30$ AU My$^{-1}$. We mention this implausible solution here for completeness, but we will not discuss it any further. The latter solution ($\delta_e = 0.14$) requires a migration rate for Saturn of $\dot{a}_6 = 4$ AU My$^{-1}$. This rate is comparable to the rates of planet migration found
in the “Jumping Jupiter” scenario proposed by Morbidelli et al. (2009) and Brasser et al. (2009). A third solution exists if we consider that eccentricities in the main belt are limited to \( \lesssim 0.35 \) due to close encounters with Mars and Jupiter. In this case, an initially Gaussian eccentricity distribution with a mean greater than \( \sim 0.3 \) would be truncated. Therefore we need only fit the lower peak of the double peak distribution at \( e = 0.09 \). Applying equation (32), we find that \( \delta_e = 0.21 \) provides a good fit. The corresponding migration rate of Saturn is \( \dot{a}_6 = 0.8 \) AU My\(^{-1}\).

We illustrate the two possible solutions for an ensemble of hypothetical asteroids having semimajor axes uniformly distributed randomly in the range 2.2 AU to 2.8 AU. In Fig. 6a the initial eccentricity distribution is modeled as a Gaussian distribution with a mean \( \langle e_i \rangle = 0.05 \) and a standard deviation of 0.01. Fig. 6b shows the eccentricity distribution after \( \nu_6 \) resonance sweeping has occurred due to the migration of Saturn at a rate of 4 AU My\(^{-1}\). The final distribution was calculated with equation (31); we used values of \( \epsilon \) shown in Fig. 2, and the value of \( \lambda \) was calculated with the aid of Fig. 1 that relates the value of \( g_p \) to the semimajor axis of Saturn. As expected, when an ensemble of asteroids with a Gaussian eccentricity distribution is subjected to the sweeping secular resonance, the result is a double-peaked eccentricity distribution. Because of the slight bias towards the upper limit of the eccentricity excitation band, proportionally more asteroids are found in the upper peak.

In Fig. 6c the initial eccentricity distribution is modeled as a Gaussian with a mean \( \langle e_i \rangle = 0.4 \) and standard deviation 0.1, but truncated at the Mars-crossing value. We used equation (31) to calculate the eccentricity distribution of this hypothetical ensemble after \( \nu_6 \) resonance sweeping due to the migration of Saturn at a rate of 0.8 AU My\(^{-1}\). Again, allowing that only those asteroids whose final eccentricities are below the Mars-crossing value will remain, the resulting post-migration eccentricity distribution is shown in Fig. 6d. In this case, we find peaks at the same eccentricity values as the peaks in the observed main belt distribution (see Fig. 5).

In both cases of possible solutions, the theoretical models yield an excess of asteroids with eccentricities greater than 0.2 than in the observed main belt. However, Minton & Malhotra (2010) showed that, on gigayear timescales, much of the \( e \gtrsim 0.2 \) population of the asteroid belt is dynamically more unstable than the \( e \lesssim 0.2 \) population. Post-sweeping dynamical erosion could result in a final eccentricity distribution resembling the observed distribution.

The estimates of Saturn’s migration rate quoted above depend strongly on the eccentricity of the giant planets during their migration. In deriving the above estimates, we adopted the present values of the giant planets’ orbital eccentricities. The \( \nu_6 \) resonance strength coefficient \( \epsilon \) (equation 3) is proportional to the amplitude of the \( p = 6 \) mode, which is related
to the eccentricities of the giant planets (namely Saturn and Jupiter). If the giant planets’ orbits were more circular than they are today at the time that the $\nu_6$ resonance was passing through the inner asteroid belt, then the derived migration timescale limits would be correspondingly longer. From equation (39), and the definition $e_j(\sin \varpi_j, \cos \varpi_j) \equiv (h_j, k_j)$, the value of $E_j^{(p)}$ is a linear combination of the eccentricities of the giant planets. Because Saturn is the planet with the largest amplitude of the $p = 6$ mode, from equation (31) the relationship between the sweep rate and the value of Saturn’s eccentricity is approximately $\lambda_{\min} \propto e_6^2$. Therefore, to increase the limiting timescale by a factor of ten would only require that the giant planets’ eccentricities were $\sim 0.3 \times$ their current value (i.e., $e_{5,6} \approx 0.015$).

6. Conclusion and Discussion

Our analysis of the $\nu_6$ sweeping indicates that the dynamical excitation of the asteroid belt during the epoch of planetesimal-driven giant planet migration was relatively short-lived. Based on the observed orbits of the population of larger asteroids ($H \leq 10.8$), we estimate that Saturn’s migration rate must have been $\gtrsim 0.15$ AU My$^{-1}$ as Saturn migrated from 8.5 to 9.2 AU. Migration rates lower than $\sim 0.15$ AU My$^{-1}$ would be inconsistent with the survival of any asteroids in the main belt. This lower limit for the migration rate of Saturn is inversely proportional to the square of the amplitude of the $g_6$ secular mode during giant planet migration. If Jupiter and Saturn’s eccentricities were $\sim 0.3 \times$ their current value (i.e., $e_{5,6} \approx 0.015$), the limit on the migration rate decreases by a factor of $\sim 10$.

The eccentricity distribution of the observed main belt may be used to set even tighter, albeit model-dependent, constraints on the migration rate of Saturn. We have identified two possible migration rates that depend on the pre-migration dynamical state of the main asteroid belt. The first is for an asteroid belt with an initial eccentricity distribution modeled as a Gaussian with $\langle e_i \rangle = 0.05$; Saturn’s migration rate of 4 AU My$^{-1}$ yields a final eccentricity distribution consistent with the observed asteroid belt. The second is for an asteroid belt with an initial eccentricity distribution modeled as a Gaussian with $\langle e_i \rangle = 0.4$, but with asteroids above the Mars-crossing value of eccentricity removed; in this case, Saturn’s migration rate of 0.8 AU My$^{-1}$ is generally consistent with the observed asteroid belt. Each of these solutions has very different implications for the primordial excitation and depletion of the main asteroid belt. The first solution, with $\dot{a}_6 = 4$ AU My$^{-1}$, would lead to little depletion of the main asteroid belt during giant planet migration, as the $\nu_6$ resonance would be unable to raise eccentricities to Mars-crossing values. This implies an initially quite dynamically cold asteroid belt with not more than roughly $\sim 2 \times$ the mass of the current main belt, consistent with dynamical erosion over the age of the solar system (Minton & Malhotra
It also implies that giant planet migration and $\nu_6$ sweeping of the main asteroid belt would not be a viable mechanism for the Late Heavy Bombardment of the inner solar system (cf. Strom et al. 2005).

The second possible solution, with $\dot{a}_6 = 0.8$ AU My$^{-1}$, would lead to substantial loss of asteroids, generally consistent with the Late Heavy Bombardment. But it also implies that the main asteroid belt was much more dynamically excited prior to resonance sweeping than we find it today. The excited asteroid belt solution also implies much greater loss of asteroids prior to sweeping, as the peak of the eccentricity distribution would be near the Mars-crossing value, and subject to strong dynamical erosion (\textendash?). This implies an “Early HeavyBombardment” of asteroids onto the terrestrial planets and the Moon in addition to a Late Heavy Bombardment once resonance sweeping occurred. This highly eccentric prior state may be consistent with excitation due to embedded embryos, as the ratio of the RMS mean eccentricity to the RMS mean sine of the inclination would be closer to 2, as predicted by theory for self excited planetesimal swarms (Wetherill 1980; Ida & Makino 1992). These solutions for the migration rate of Saturn depend strongly on the eccentricities of the giant planets during their migration. If the giant planets’ eccentricities were $\sim 0.3 \times$ their current value (i.e., $e_{5,6} \approx 0.015$), the limits on the migration timescale increase by a factor of $\sim 10$. These limits also depend strongly on the pre-sweeping dynamical state of the main asteroid belt.

In order to elucidate the effects of $\nu_6$ resonance sweeping, we have made a number of simplifying assumptions to arrive at an analytically tractable model. These simplifications include neglecting the effects of planets other than Jupiter and Saturn, the effects of sweeping jovian mean motion resonances on asteroids, the effects of a presumed massive Kuiper belt during the epoch of planet migration, and the self-gravity and collisional interactions of a previously more massive asteroid belt. In addition, our analysis was carried out in the planar approximation, thereby neglecting any eccentricity-inclination coupling effects. These neglected effects can be expected to reduce somewhat the lower limits on Saturn’s migration speed that we have derived, because, in general they would reduce the effectiveness of the $\nu_6$ in exciting asteroid eccentricities. Perhaps more importantly, giant planet migration would also lead to the sweeping of the main asteroid belt by the $\nu_{16}$ inclination secular resonance (Williams & Faulkner 1981) whose effects could be used to infer additional constraints; this is beyond the scope of the present paper but will be addressed in future work.

A number of other studies have derived limits on the speed of planetesimal-driven giant planet migration. Murray-Clay & Chiang (2005) exclude an $e$–folding migration timescale $\tau \leq 1$ My to 99.65% confidence based on the lack of a large observed asymmetry in the
population of Kuiper belt objects in the two libration centers of the 2:1 Neptune mean motion resonance. Boué et al. (2009) exclude $\tau \leq 7$ My based on the observed obliquity of Saturn. These lower limits on the migration timescale are slightly incompatible with the lower limit on the rate of Saturn’s migration of $\dot{a}_6 > 0.15$ AU My$^{-1}$ we derive based on the inner asteroid belt distribution. One way these can be reconciled with our results is if Saturn’s orbital eccentricity were a factor $\sim 2$ smaller than its present value as it migrated from 8.5 AU to 9.2 AU; then, some mechanism would need to have increased Saturn’s eccentricity up to its present value by the time that Saturn reached its present semimajor axis of $\sim 9.6$ AU.

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A. Fitting the eccentricity and inclination distributions

The binned eccentricity and inclination distributions may be modeled as Gaussian probability distribution function, given by:

$$p(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(x - \mu)^2}{2\sigma^2} \right],$$

(A1)

where $\sigma$ is the standard deviation, $\mu$ is the mean, and $x$ will be either $e$ or $\sin i$. With an appropriate scaling factor, equation (A1) can be used to model the number of asteroids per eccentricity bin. However, rather than fit the binned distributions directly, we instead perform a least squares fit of the unbinned sample to the Gaussian cumulative distribution function given by:

$$P(x) = \frac{1}{2} + \frac{1}{2} \text{erf} \left( -\frac{x - \mu}{\sigma \sqrt{2}} \right).$$

(A2)

For the eccentricity distribution the best fit parameters are:

$$\mu_e = 0.135 \pm 0.00013,$$

$$\sigma_e = 0.0716 \pm 0.00022.$$

A better fit to the eccentricity distribution can be obtained using a double-peaked, symmetric, Gaussian distribution, where:

$$p'(x) = \frac{A'}{\sigma' \sqrt{2\pi}} \left\{ \exp \left[ -\frac{(x - \mu'_1)^2}{2\sigma'^2} \right] + \exp \left[ -\frac{(x - \mu'_2)^2}{2\sigma'^2} \right] \right\}.$$  

(A3)
The cumulative distribution function for equation (A3) is

\[ P'(x) = \frac{1}{2} + \frac{1}{4} \left[ \text{erf}\left( -\frac{x - \mu'_1}{\sigma \sqrt{2}} \right) + \text{erf}\left( -\frac{x - \mu'_2}{\sigma \sqrt{2}} \right) \right]. \] (A4)

We performed a least squares fit of eccentricity distribution to equation (A4) and obtain the following best-fit parameters:

\[ \mu'_{e,1} = 0.0846 \pm 0.00011, \]
\[ \mu'_{e,2} = 0.185 \pm 0.00012, \]
\[ \sigma'_e = 0.0411 \pm 0.00020. \]

We evaluated the goodness of fit using the Kolmogorov-Smirnov (K-S) test. The K-S test determines the probability that two distributions are the same, or in our case how well our model distributions fit the observed data (Press et al. 1992). The K-S test compares the cumulative distribution of the data against the model cumulative distribution function. In the case of the eccentricity distribution, observed asteroid data has a probability of \(4.5 \times 10^{-2}\) that it comes from the best fit Gaussian distribution given by equation (A2), but a probability of 0.73 that it comes from the double-peaked Gaussian given by equation (A4).
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Fig. 1.— a) The $g_6$ eigenfrequency as a function of Saturn’s semimajor axis, for Jupiter fixed at 5.2 AU. The dashed line shows the result from linear secular theory, with a correction for the effect of the near 2:1 mean motion resonance between Jupiter and Saturn (Malhotra et al. 1989). The solid line shows the result from numerical spectral analysis of 233 solar system integrations (see text for explanation). The locations of Jupiter-Saturn MMRs which have an effect on the value of $g_6$ are indicated by vertical dotted lines. b) The location of the $\nu_6$ resonance (at zero inclination) as a function of Saturn’s orbit. The frequencies $g_6$ and $g_0$ were calculated for each value of Saturn’s semimajor axis, $a_6$, and then the location $a_{\nu_6}$ was determined by finding where $g_0 - g_6 = 0$. The dashed line shows the result from linear secular theory, with a correction for the effect of the 2:1 near-MMR between Jupiter and Saturn (Malhotra et al. 1989). The solid line was obtained by using the $g_6$ eigenfrequencies obtained from spectral analysis of the 233 numerical integrations, as shown in (a).
Fig. 2.— The value of the coefficient $\epsilon$ defined by equation (3) as a function of the zero inclination location of the $\nu_6$ resonance. The values of $E^{(i)}_j$ were calculated using first order Laplace-Lagrange secular theory with corrections arising from the 2:1 Jupiter-Saturn mean motion resonance.
Fig. 3.— Comparison between the numerical solution of the averaged equations (equations 9–10) and full n-body numerical integrations of test particles at 2.3 AU. The dashed lines represent the envelope of the predicted final eccentricity, equation (32). The values of $\lambda$ given are in the canonical unit system described in $\S$2.
Fig. 4.— The final eccentricity distribution of an ensemble of particles with initial eccentricity $e_i = 0.1$ and uniformly distributed values of the phase angle $\varpi_i$. The effect due to the sweeping $\nu_6$ resonance was modeled using equation (31). The parameters were chosen to simulate asteroids at $a = 2.3$ AU, and with $\dot{a}_0 = 1$ AU My$^{-1}$. 
Fig. 5.— Proper eccentricity distribution of observed asteroids with absolute magnitude $H \leq 10.8$, excluding members of collisional families. The proper elements were taken from the AstDys online data service (Knežević & Milani 2003). Family members identified by Nesvorný et al. (2006) were excluded. The solid lines are the best fit Gaussian distribution to the observational data. The dashed line is the best fit double-peaked distribution.
Fig. 6.— The effects of the sweeping $\nu_6$ resonance on an ensemble of asteroids with semimajor axes 2.2–2.8 AU and a uniform distribution of pericenter longitudes. (a) Initial distribution of eccentricities where the mean of the distribution is 0.05 and the standard deviation is 0.01. (b) The final distribution of eccentricities after $\nu_6$ sweeping, using the analytical result given by equation (31), for $\dot{a}_6 = 4.0$ AU My$^{-1}$. The two peaks in the final eccentricity distribution are at approximately the same values as the observed peaks in the main asteroid belt eccentricity distribution shown in Fig. 5. (c) Initial distribution of eccentricities where the mean of the distribution is 0.40 and the standard deviation is 0.1. Asteroids with eccentricities greater than the Mars-crossing value were removed. (d) The final distribution of eccentricities after $\nu_6$ sweeping for $\dot{a}_6 = 0.8$ AU My$^{-1}$. The ordinates in the four panels are not to the same scale.
Fig. 7.— Estimated final eccentricity of asteroids as a function of asteroid semimajor axis and eccentricity for three different migration rates of Saturn using equation (32). Asteroids swept by the $\nu_6$ resonance can have a range of final eccentricities depending on their apsidal phase, $\varpi_i$. The outermost shaded region demarcates the range of final eccentricities for asteroids with an initial eccentricity $e_i = 0.4$. The innermost shaded region demarcates the range of final eccentricities for asteroids with an initial eccentricity $e_i = 0.2$. The solid line at the center of the shaded regions is the final eccentricity for an asteroid with an initial eccentricity $e_i = 0$. 