PTYS 411 – Geology and Geophysics of the Solar System
Homework #1 - Solutions

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1) Freefall timescale for a giant molecular cloud

A particle in freefall is on an orbital trajectory. Since the fall starts from rest the particle can be considered to be starting at apoapse (furthest distance in the orbit) and orbiting on a near-parabolic trajectory where apoapse distance ~ twice the semimajor axis (a). Falling from apoapse to periapse (almost the same as into the central body) takes half an orbital period. Kepler's 3rd law relates the semi-major axis to the orbital period.

$$P^2 = \frac{4\pi^2}{GM_*}a^3 \quad so \quad \frac{P}{2} = \sqrt{\frac{\pi^2 a^3}{GM_*}}$$

where G is the gravitational constant (6.67×10^{-11}). The attracting mass in a gravitational system is only the mass closer to the center-of-mass than the gas packet we're interested in. Since we're assuming the cloud has a constant density (ρ) and the initial distance of our gas-packet is (2.a) then:

$$M_* = \frac{4}{3}\pi(2a)^3\rho = \frac{32\pi}{3}a^3\rho$$

$$so: \quad \frac{P}{2} = \sqrt{\frac{\pi^2 a^3}{G}\frac{3}{32\pi a^3 \rho}} = \sqrt{\frac{3\pi}{32G\rho}}$$

This is the freefall timescale.

How does this timescale compare to the age of the solar system?

A typical density of a giant molecular cloud is a few 1000 H_2 molecules per cubic centimeter, which is ~10^{-17} kg m^{-3}. Plugging in the numbers we find the free-fall timescale to be 2.1×10^{13}seconds or about 677 Kyr. This is very short in comparison with the solar system's age (4.5 Gyr); cloud collapse is a fast process.

2) Show that the total energy delivered when building a planet by impacting planetesimals is: $\dfrac{3GM_E^2}{5R_E}$

A body falling from infinity, which was initially at rest has zero energy. Energy is conserved, so when it hits the planet its kinetic and potential energies add up to zero. In this case, the kinetic energy is the negative of the potential energy. If an impactor delivers a mass Δm then the kinetic energy it delivers (ΔE) is given by:

$$\Delta E = \frac{GM\ \Delta m}{R}$$

where R and M are the radius and mass of the planet so far. We assume constant density so: $M=(4/3)\pi R^3\rho$. The incoming mass, Δm, is spread over the surface area of the growing planet causing a radius increase ΔR. Again we assume constant density so: $\Delta M=4\pi R^2\ \rho\ \Delta R$. Combining these relationships:

$$\Delta E = \frac{G\left[\frac{4}{3}\pi R^3\rho\right]\left[4\pi R^2\rho\Delta R\right]}{R}$$

$$rearrange: \quad \Delta E = 3G\left(\frac{4}{3}\pi\rho\right)^2\ R^4\ \Delta R$$

We need to integrate this from a radius of 0 to R_p (final planetary radius):

$$E = 3G\left(\frac{4}{3}\pi\rho\right)^2 \int_0^{R_p} R^4\ \delta R$$

$$E = \frac{3G}{5}\left(\frac{4}{3}\pi\rho\right)^2\left(R_P^5 - 0\right) = \frac{3G}{5R_P}\left(\frac{4}{3}\pi\rho R_P^3\right)^2$$

The final mass of the planet is given by: $M_P = \frac{4}{3}\pi\rho R_P^3$ so the total energy is:

$$E = \frac{3GM_P^2}{5R_P}$$

Assume all this energy goes into heat, write an expression for the temperature rise of the planet?

The relationship between energy and temperature rise is given by: $E = C_P M_P\ \Delta T$, where C_p is the specific heat capacity and M_p is the planetary mass. Incorporating the above expression:

$$E = \frac{3GM_P^2}{5R_P} = C_P M_P \,\Delta T$$

$$\Delta T = \frac{3GM_P}{5R_P C_P}$$

For the Earth, ΔT=46664 K
This will easily be enough to melt all the material.

Give a reason why this temperature rise might be an overestimate and a reason that it might be an underestimate.

It's an overestimate because in reality the planet has time to radiate away heat i.e. all this material does not arrive instantaneously.
It's an underestimate because we assume particles start at infinity and at rest (zero net energy). This is the minimum energy scenario. Most particles will have some additional kinetic energy of their own to deliver.

What is the minimum size a planet must grow to in order to be completely melted by this process? Compare this to the size of the terrestrial planets, the Moon and some of the largest asteroids. Should we expect magma oceans on newly formed planets?

Rewrite the above temperature difference equation in terms of radius:

$$\Delta T = \frac{3GM_P}{5R_P C_P} = \frac{3G}{5R_P C_P}\left(\tfrac{4}{3}\pi\rho\right)R_p^3$$

$$\Delta T = \frac{4G\pi\rho}{5C_P}R_p^2 = 6.28\times 10^{-10}R_p^2$$

If the body just about melts then ΔT~1000K. In that case, R_P = 1261 km. If R_P is greater than this then the body's temperature will be high enough to melt due solely to its accretional energy.

This radius is smaller than that of the terrestrial planets and Earth's Moon so we'd expect these bodies to have melted and to have processed magma oceans in the past.

Conversely, this radius is larger than all the current asteroids so we don't expect those bodies melted entirely by this mechanism and they did not have a magma ocean in the past. (However, large asteroids can still partly melt and differentiate and heat from radioactive decay becomes important as well for these bodies.)

3) **Isostasy. On Venus plate tectonics is absent. Down-welling flows in Earth's mantle are usually associated with subduction whereas on Venus it's thought to cause shortening of the crust.**
   Think of a linear strip of the crust that has a width w_o. It gets compressed and reduced in width to w. i.e. the compression factor is $C_f = w_o/w$. This compression builds mountains that are supported by Airy isostasy i.e. they float like icebergs in the Venusian mantle. Show that the mountain height is given by

$$h = T_L \frac{\rho_m - \rho_c}{\rho_m}\left(C_f - 1\right)$$

   where the mantle and crust densities are ρ_m (~3300 kg m^{-3}) ρ_c (~2750 kg m^{-3}) respectively and T_L is the thickness of the crust. How tall do these mountains get when crustal rocks get compressed by a factor of two?

There are two useful relations to derive here, conservation of volume and balance between the weight and buoyancy forces. For conservation of volume, the crustal strip originally has a volume (per unit length along the strip) of $T_L w_o$. After the compression the crust has mountains (height h) and a root sticking into the mantle (height h_r), giving a volume (per unit length along the strip) of $(T_L+h+h_r)w$, where w is the new, shortened width.

$$w_o\left(T_L\right) = w\left(T_L + h + h_r\right)$$

$$\left(\frac{w_o}{w} - 1\right)T_L = h + h_r$$

$$h\left(1 + \frac{h_r}{h}\right) = T_L\left(C_f - 1\right)$$

$$h = T_L\left(1 + \frac{h_r}{h}\right)^{-1}\left(C_f - 1\right)$$

Now think of the balance between buoyancy forces and weight. The crust usually has these forces in balance so we need only think about the extra masses i.e. the mountains above the surface and the root below. The weight of the mountains and root (again, per unit length along the lithospheric strip) are $(wh\rho_c g)$ and $(wh_r \rho_c g)$ respectively, whereas the buoyancy force from the displaced mantle material is $(wh_r \rho_m g)$. Equating these gives:

$$wh\rho_c g + wh_r \rho_c g = wh_r \rho_m g$$

$$\rho_c h + h_r \rho_c = h_r \rho_m$$

$$h = h_r\left(\frac{\rho_m - \rho_c}{\rho_c}\right)$$

$$\frac{h_r}{h} = \left(\frac{\rho_c}{\rho_m - \rho_c}\right)$$

Combining these two relations:

$$h = T_L \left(1 + \frac{\rho_c}{\rho_m - \rho_c}\right)^{-1} (C_f - 1)$$

$$h = T_L \left(\frac{\rho_m}{\rho_m - \rho_c}\right)^{-1} (C_f - 1)$$

$$h = T_L \left(\frac{\rho_m - \rho_c}{\rho_m}\right) (C_f - 1)$$

Assume a representative venusian crustal thickness of 70km

When C_f =2, ρ_m ~3300 kg m^{-3} ρ_c ~2750 kg m^{-3} then h=0.17 T_L. If we take a representative crustal thickness of 70km then h=11.6km.

This is close to the maximum height of mountains on Venus and so extreme crustal shortening (x2 is a lot) is required if no other mechanisms are operating.

4) The very large lunar south pole Aitken basin is currently about 8km deep and has no major gravity anomaly associated with it. If this impact originally excavated all the way through the crustal material (density 2800 kg m^{-3}) to the mantle (3300 kg m^{-3}) then how thick is the lunar crust?

The mantle rebounded somewhat to fill the crater. As the mantle is denser it doesn't need to fill up the impact basin to produce the same gravity signature. Think of two columns, one in the crater and one outside. There's no gravity anomaly, so they contain the same amount of mass.

$$\rho_c g T_L = \rho_m g (T_L - 8km)$$

$$1 - \frac{\rho_c}{\rho_m} = \frac{8km}{T_L} \quad or \quad T_L = 8km\left(\frac{\rho_m}{\rho_m - \rho_c}\right)$$

Plugging in the numbers gives a crustal thickness of ~53km. If there were some crustal material left in the basin then we need to add that to the final answer e.g. if there were still 7km of crustal material in the crater above the mantle then the crustal thickness would be 60km not 53km.

5) Define the difference between a planet's lithosphere and crust.
The crust is the upper layer of rock that is compositionally distinct from the planet's mantle. The lithosphere is the upper layer of rock on a planet that mechanically behaves more like an elastic solid than flowing liquid.

What sets the thickness of a planets lithosphere? Is the lithosphere of the Earth thicker or thinner than its crust (let's just assume we're talking about oceanic crust here)?
The mechanical behavior of a given solid is primarily controlled by its temperature. As temperature increases with depth the rocky material starts to behave in a more fluid-like manner as less like an elastic solid. There's no exact value for the base of the lithosphere (as it depends on how long you want elastic stresses to be supported for), but a useful rule of thumb is that the lithosphere ends when temperatures reach half the melting point.

Is the lithosphere on small planets like the Moon and Mars thicker or thinner than larger planets like Earth and Venus? Why?
The lithosphere of small planets is thicker because they have cooled off more so one must go to greater depths before reaching a temperature of about half the melting point. These smaller planets have cooled off more because the have greater surface area per unit volume.

How does the behavior of surface and deeply buried rocks differ when they're put under stress?
Rocks under high pressure deform in a ductile rather than brittle way. i.e. they permanently change shape without developing macroscopic fractures localized in one spot. Instead, this shape-change is accommodated by many microscopic fractures evenly spread throughout the material.

What's the pressure at the bottom on the Marineris trench on the Earth (~11km below sea level)? If I were to put a rock down there what differential stress would it experience? Would you expect typical rocks to fracture in this environment?
The pressure is the weight of the overlying material per unit area i.e. density (1000 Kgm^{-3} for water) * thickness (11000m) * gravity (9.8 ms^{-2}) = 108 MPa (there's also a small contribution from the weight of the overlying atmosphere of 0.1 MPa).

The differential stress is zero though as this pressure acts equally in all directions! So, no rocks will not fracture because they are on the seafloor – even though the pressures are huge.

6) Show that the energy production per unit mass (H) is: $H = {}^{3\,q}\!\big/_{R_P\,\rho}$ where q is the surface heat flux. q was measured by the Apollo astronauts in two locations at 16 and 21 mW m^{-2}, what values do you get for H.

The heat production rate for the entire body is H times its mass. All this heat must be conducted out though its surface so the heat flux through 1 m^2 is:

$$q = \frac{H\ M_P}{4\pi R_p^2}$$

$$q = \frac{H\ {}^4\!/_3\,\pi R_P^3 \rho}{4\pi R_p^2} = \frac{H\ R_P \rho}{3}$$

$$so: \quad H = \frac{3q}{R_P \rho}$$

Using the average of those heat flux values with a lunar radius and an average density of 3300 kg m^{-3}, we find H=9.68x10^{-12} J kg^{-1}

The heat flux (F) inside the Moon from conduction is: $F = -k\frac{\delta T}{\delta R}$ where k is the thermal conductivity. Show that the temperature (T) inside the Moon is: $T = T_o + \frac{\rho H}{6k}\left(R_P^2 - R^2\right)$.

Think of a thin spherical shell within the body. All the heat produced within this shell must be conducted out though it. So, as in the first section of the problem:

$$F = \frac{H\ M(R)}{4\pi R^2}$$

$$F = \frac{H\ {}^4\!/_3\,\pi R^3 \rho}{4\pi R^2} = \frac{H\rho R}{3}$$

This flux is the conductive heat flux. Equate these and Integrate between some interior radius, R, (temperature T) and the surface (R=R$_p$ and T=T$_0$).

$$-k\frac{\delta T}{\delta R} = \frac{H\rho R}{3}$$

$$\int_{T_o}^{T} \delta T = -\frac{H\rho}{3k}\int_{R_P}^{R} R\,\delta R$$

$$T - T_o = -\frac{H\rho}{6k}\left(R^2 - R_P^2\right)$$

$$or: T = T_o + \frac{H\rho}{6k}\left(R_P^2 - R^2\right)$$

What is the temperature at the center of the Moon? What's the main thing wrong with this model?

Combining this with the expression for H at the beginning of the question and setting R to 0, we find.

$$T_c = T_o + \frac{\rho}{6k}\frac{3q}{R_P\rho}\left(R_P^2 - 0^2\right)$$

$$T_c = T_o + \frac{qR_P}{2k}$$

Using typical values for k (~2 $JK^{-1}s^{-1}m^{-1}$) we find T_c is 8334K.

Show that the pressure within the Moon is given by: $P = \frac{2\pi}{3}G\rho^2\left(R_P^2 - R^2\right)$

Consider a spherical shell within the body of radius R, thickness ΔR and mass Δm. The pressure it exerts is its weight divided by its surface area:

$$\Delta P = -\frac{g(R)\,\Delta m}{4\pi R^2}$$

where g(R) is the acceleration due to gravity caused by the mass of material interior to the shell. Expanding the above expression:

$$\Delta P = -g(R)\frac{\Delta m}{4\pi R^2}$$

$$\Delta P = -\frac{G\,{}^4\!/_3\,\pi R^3\rho}{R^2}\frac{4\pi R^2\Delta R\rho}{4\pi R^2}$$

$$\Delta P = -\frac{4\pi G\rho^2}{3}R\,\Delta R$$

Integrate this between some interior radius, R, (pressure P) and the surface ($R=R_p$ and P=0).

$$\int_0^P \delta P = -\frac{4\pi G\rho^2}{3}\int_{R_P}^{R} R\,\delta R$$

$$P = -\frac{2\pi G\rho^2}{3}\left(R^2 - R_p^2\right)$$

$$P = \frac{2\pi G\rho^2}{3}\left(R_p^2 - R^2\right)$$

Look at the olivine phase diagram at this pressure what temperature do you need to have a molten core?

According to the relation above, the central pressure in the Moon is given by:

$$P_c = \frac{2\pi}{3} G\rho^2 R_P^2$$

Substituting in the lunar radius and an average lunar density of 3300 kg m^{-3}, we find P_c to be 4.6 GPa. By inspecting the olivine phase diagram, we find melting occurs, at that pressure at temperatures of ~1500-1900K.

Given the pressure and temperature we calculated for the center of the Moon in this question we'd expect a molten core.

The main problem with this model is that is assumes that the heat producing radioactive elements are spread evenly throughout the body whereas in reality they are concentrated in the crust. Temperature in the Moon's interior stops increasing with depth after it gets to ~1500K.