PTYS 411 – Geology and Geophysics of the Solar System
Homework #2 – Assigned 2/2, due 2/16

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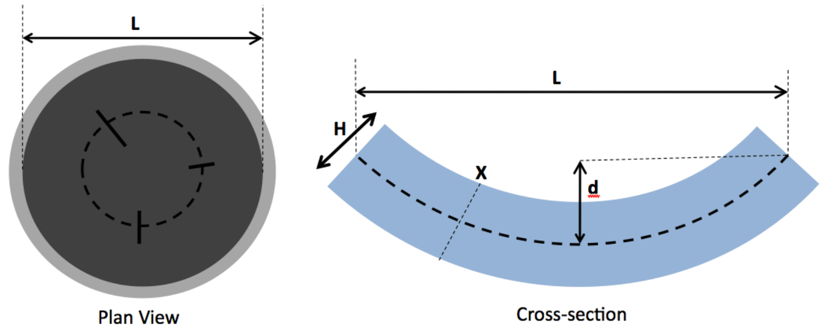
1) **The subsidence of lunar basins under the weight of the mare basalts has generated thrust faults near their center. Imagine a circular plate, thickness H, has been downwarped a distance d. Its top surface is compressed while the material at a depth of H/2 feels zero strain (see cartoon below).**



Show that the circumferential surface stresses (parallel to the rim) at point X (half way between center and rim) are given by:

$\frac{4dHE}{L^2}$ **where E is Young's Modulus (you can look this up)**

Imagine a center of curvature above the mare with a radius r (distance to the neutral sheet in the mare center). By pythagorous:

$$(r-d)^2 + \left(\frac{L}{2}\right)^2 = r^2 \quad so \quad r = \frac{L^2}{8d}\left(1 + \frac{4d^2}{L^2}\right)$$

$$if \quad d << L \quad then \quad r \cong \frac{L^2}{8d}$$

The strain is the change in the circumference of the horizontal circle passing through X:

$$\varepsilon = \frac{2\pi r\theta - 2\pi\left(r - H/2\right)\theta}{2\pi r\theta} = \frac{H}{2r}$$

$$substitute \ for \ r: \quad \varepsilon = \frac{4dH}{L^2}$$

where θ is the angle between the lines connecting the center of curvature to the center of the mare and to the point X. The stress is Young's modulus times the strain:

$$\sigma = \frac{4dH}{L^2}E$$

Try this for one of the mare where L is ~300km, H is ~50km (lithosphere at the time of loading) and d is ~2km. Is the resultant stress large enough to overcome typical rock strengths? (Assume d << L in this problem)

A typical Young's modulus for basalt is 70 GPa. Substituting these numbers into the approximation above yields stresses of 310 MPa. The compressive strength of basalt is on the order of 10^8 Pa so this stress should be enough to fault the mare.

If three thrust faults form with a typical dip then how much displacement will each fault experience?

For this we need to know the actual amount of shrinkage, not just the strain. The original radius of the circle passing through X of the mare is $r\theta$. So the circumference (C) is $2\pi r\theta$ and the change in circumference (ΔC) is this times the strain, substituting for r and ε gives $\Delta C=\pi H\theta$. We know that $\sin(2\theta)$ is L/2r, which is 4d/L. d/L is small so $\sin(2\theta)$ is ~2θ. So θ~2d/L and $\Delta C=2\pi dH/L$.

If there are three faults then each on has a horizontal displacement of $\Delta C/3$. If we assume a typical fault dip of about 30 degrees then each one has a total displacement of $\Delta C/3/\cos(30)$. Plugging in the numbers gives fault displacements of ~800m.

2) ***Moments of inertia*. A lot can be discovered from a planet from its moment of inertia. Moment of inertia depends on the geometry of the object sphere vs empty shell vs point etc… but in general is given by k M R^2, where M is the mass, R is the radius and k is a constant e.g. for a point of mass M orbiting at distance R k=1, for a rotating thin hollow sphere M is the mass of the shell, R is its radius and k = 2/3.**

    Use the moment of inertia of the thin hollow shell mentioned above to show that the moment of inertia of a homogeneous solid (and spherical) planet is $\frac{2}{5} M_P R_P^2$

The moment of inertia of a thin hollow shell is given by: $\frac{2}{3} M_{SHELL} R_{SHELL}^2$. We can construct the moment of inertia of a solid sphere (mass M_P) by adding together a lot of thin shells (mass Δm, radius R, thickness ΔR).

$$\Delta I = \frac{2}{3} \Delta m R^2 = \frac{2}{3}\left(4\pi R^2 \Delta R \rho\right) R^2$$

$$I = \int \Delta I = \int_0^{R_P} \frac{2}{3}\left(4\pi R^2 \Delta R \rho\right) R^2$$

$$I = \frac{2}{3} 4\pi\rho \int_0^{R_P} R^4 \Delta R = \frac{2}{3} 4\pi\rho \frac{R_P^5}{5}$$

$$I = \frac{2}{5}\left(\frac{4}{3}\pi R_p^3 \rho\right) R_p^2 = \frac{2}{5} M_P R_p^2$$

Show that for a differentiated planet (radius R_p with density ρ_c in a core of radius R_c and density ρ_m in the mantle surrounding the core) the moment of inertia is:

$$I_{differentiated} = \frac{2}{5}\left[\frac{1 + c\,x^5}{1 + c\,x^3}\right] M_P R_P^2$$

$where \quad c = \frac{(\rho_c - \rho_m)}{\rho_m} \quad and \quad x = \frac{R_c}{R_P}$. **The derivation isn't that bad. Break the previous integral into two parts.**

As before, we add up the moment of inertia from spherical shells via an integral, but now we have two different values of density for different ranges of R so we split the integral into two parts.

$$I = \int_0^{R_C} \tfrac{2}{3}\left(4\pi R^2 \Delta R \rho_C\right) R^2 + \int_{R_C}^{R_P} \tfrac{2}{3}\left(4\pi R^2 \Delta R \rho_m\right) R^2$$

$$I = \tfrac{2}{3} 4\pi \left(\rho_C \frac{R_C^5}{5} + \rho_m \frac{\left(R_P^5 - R_C^5\right)}{5} \right)$$

$$I = \tfrac{2}{5} \tfrac{4}{3} \pi \left(\rho_m R_P^5 + \left(\rho_C - \rho_m\right) R_C^5 \right)$$

$$I = \tfrac{2}{5} \tfrac{4}{3} \pi \rho_m R_P^5 \left(1 + \frac{\left(\rho_C - \rho_m\right)}{\rho_m} \frac{R_C^5}{R_P^5} \right)$$

$$I = \tfrac{2}{5} \left(\tfrac{4}{3} \pi \rho R_P^3 \right) R_P^2 \frac{\rho_m}{\rho} \left(1 + \frac{\left(\rho_C - \rho_m\right)}{\rho_m} \frac{R_C^5}{R_P^5} \right)$$

$$I = \tfrac{2}{5} M_P R_P^2 \left(\frac{\rho_m}{\rho} \right) \left(1 + c x^5 \right) \quad where \quad c = \left(\rho_c - \rho_m\right) \big/ \rho_m \quad and \quad x = R_c \big/ R_P$$

To finish the problem we need an expression for the ratio of the mantle density (ρ_m) and the mean density (ρ). The mean density is given by the volume-weighted average:

$$\rho = \left[\frac{\left(\tfrac{4}{3}\pi R_C^3\right)}{\left(\tfrac{4}{3}\pi R_P^3\right)} \right] \rho_C + \left[\frac{\left(\tfrac{4}{3}\pi R_P^3 - \tfrac{4}{3}\pi R_C^3\right)}{\left(\tfrac{4}{3}\pi R_P^3\right)} \right] \rho_m$$

$$\frac{\rho}{\rho_m} = x^3 \frac{\rho_C}{\rho_m} + \left[1 - x^3\right] = 1 + \left[\frac{\rho_C}{\rho_m} - 1 \right] x^3 = 1 + c x^3$$

$$so: \frac{\rho_m}{\rho} = \frac{1}{1 + c x^3}$$

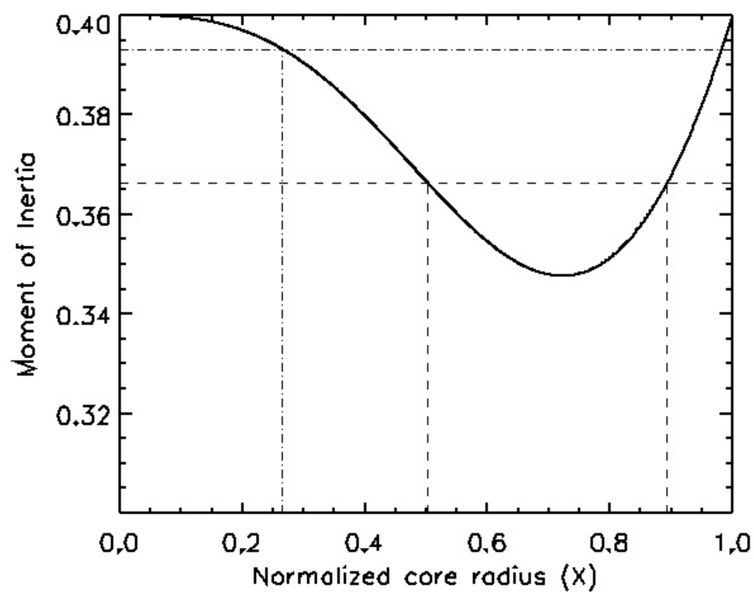
Substituting this back into our expression for I, we find:

$$I = \tfrac{2}{5} M_P R_P^2 \frac{1}{1 + c x^3} \left(1 + c x^5\right)$$

$$I = \tfrac{2}{5} \left[\frac{1 + c\, x^5}{1 + c\, x^3} \right] M_P R_P^2$$

Assume that the core is twice as dense as the mantle. Plot the value of the geometry-dependant constant $\tfrac{2}{5}\left[\frac{1 + c\, x^5}{1 + c\, x^3}\right]$ vs. x.

In the case where the core is twice as dense as the mantle then c=1.



Mars has a moment of inertia of 0.3662 MR^2. Use your plot to find x, and by extension the core size on Mars (the real value core radius has been estimated at ~0.48 R_P)?

As can be seen from the above plot the two possible solutions (dashed lines) for the Martian moment of inertia at x=0.503 and x=0.892. Leading to core sizes of 0.503 R_p= 1708 km and 0.892 R_p= 3029 km.
Although either of these solutions could explain the moment of inertia each one implies a very different planetary mass when you assume sensible values for density of the core and mantle.
Working backwards, you can use one of the values of x that we calculated along with the known total mass of the planet (and still using the assumption that c=1) to get the values of density for the core and mantle. These will either be sensible or not depending on whether you picked the right solution for x.

There are two solutions. Although it's clear which is the correct one, how would you distinguish between them if it wasn't so clear? Give a hand-waving explanation as to why are there are two solutions?

An undifferentiated body with no core has a moment of inertia of 0.4. As the body becomes progressively more differentiated the moment of inertia is reduced. However, as the core gets larger and approaches the total size of the object the moment of inertia rises again, as the body is becoming more homogeneous. When the core occupies the entire body you're back where you started from with a value of 0.4. Moment of inertia only measures the central concentration of mass not the amount of mass so a case with zero core has the same value as a case with 100% core as they're both homogeneous objects.

There is a minimum value of moment of inertia with an associated core size. For the higher values of moment of inertia you could have a core that is larger or smaller than this 'special' size.

The Moon has a moment of inertia of 0.3931 MR^2. Use your plot to find x, and by extension the lunar core size?

Using the plot above (dot-dash lines) we see that X = 0.265. The lunar core is 0.265*R_{moon}= 461km.

If Mercury's core radius is 0.72 of its total radius and its moment of inertia is 0.33 MR^2 then what is the density ratio between its core and mantle?

Rearranging the above expression for 'I' of a differentiated planet:

$$0.33 = \tfrac{2}{5}\left[\frac{1 + c\,x^5}{1 + c\,x^3}\right]$$

$$0.825\left(1 + c\,x^3\right) = 1 + c\,x^5$$

$$1 - 0.825 = \left(0.825x^3 - x^5\right)c$$

$$c = \frac{0.175}{\left(0.825x^3 - x^5\right)}$$

$$\frac{\rho_C - \rho_m}{\rho_m} = \frac{0.175}{\left(0.825x^3 - x^5\right)}$$

$$\frac{\rho_C}{\rho_m} = 1 + \frac{0.175}{\left(0.825x^3 - x^5\right)}$$

If x=0.72 then c=1.53 so the core is 2.53 times as dense as the mantle.

Titan's mean density is 1880 kg m^{-3}, assume that it's made up only of different phases of water (~1000 kg m^{-3}) and rock (~3300 kg m^{-3}) and that it's fully differentiated. What moment of inertia factor do you expect Titan to have? Cassini tracking data published last year has shown this value to be 0.34. Compare this number to what you expected from the above calculation. What do we learn about Titan from this comparison?

The volume fraction of rock is $\dfrac{V_{sil}}{V_T} = \left(\dfrac{\bar{\rho} - \rho_{ice}}{\rho_{sil} - \rho_{ice}}\right)$.

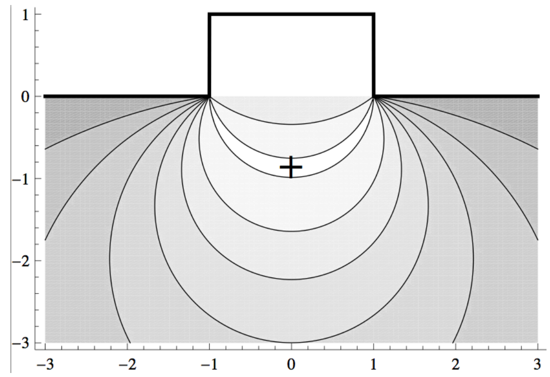
Given the densities quoted in the question this fraction is 0.38. We're assuming that Titan is fully differentiated so this rocky material should be fully concentrated in a core. The size of the core as a fraction of the planetary radius is the cube root of this volume fraction i.e. the rocky core has a radius 0.726 times that of Titan.

Using x=0.726 and c=2.3 in the above equations implies that I = 0.3114 MR^2

Titan's moment of inertia factor turns out to be 0.34, indicating that its core is not differentiated or that the core is made up of low-density hydrated silicates.

3) ***Io's mountains***

We discussed in class the maximum shear stress generated by a surface load is just a fraction (usually about a third to a half) of the peak load itself. For example, the rectangular block mountain and stress contours shown schematically here generates a peak shear stress in the subsurface of 0.352 ρgh at a depth of 0.865w (w=mountain width which we'll assume to be equal to h for now) (Melosh 2011). For the mountain to be supported then this stress must be less than the typical strength of rocks (~100 MPa).



Show that the maximum topography than can be supported like this is:

$$\boldsymbol{h \approx \left(\frac{0.7\ \ \sigma_y}{G\bar{\rho}\ \rho_c}\right) \Big/ R_{planet}}$$

Where $\bar{\rho}\ \rho_c$ are the planet's mean density and crustal density respectively and σ_y is the strength of rock.

The shear strength of the rock must be more than the peak shear stress (0.352 $\rho_c$gh) for the mountain to be supported. For the highest possible mountain then these quantities are just equal.

$$\boldsymbol{\sigma_y} = 0.352\rho_c gh$$

Gravitational acceleration (g) depends on the mass and size of the planet i.e. $\boldsymbol{g} = \frac{GM}{R^2}$

Combining and rearranging:

$$\boldsymbol{h = \left(\frac{0.352\ \ \sigma_y\ \ R^2{}_{planet}}{GM\ \rho_c}\right)}$$

Replace the mass of the planet (M) with the volume times the mean density ($\boldsymbol{\bar{\rho}}$)

$$\boldsymbol{h = \left(\frac{0.352\ \ \sigma_y\ \ R^2{}_{planet}}{G\ {}^4\!/_3\,\pi\ R^3{}_{planet}\ \ \bar{\rho}\ \ \rho_c}\right)}$$

$$\boldsymbol{h = \left(\frac{0.352\ {}^3\!/_{4\pi}\ \ \sigma_y}{G\bar{\rho}\ \rho_c}\right) \Big/ R_{planet}}$$

$$\boldsymbol{h \approx \left(\frac{0.7\ \ \sigma_y}{G\bar{\rho}\ \rho_c}\right) \Big/ R_{planet}}$$

In class we discussed how well (or not) this works for the terrestrial planets. Using the above relationship, how high are the highest mountains on Jupiter's moon Io predicted to be? (crustal density is ~3000 Kg m^{-3})

Plugging in properties for Io i.e. $\bar{\rho}$ 3530 Kg m^{-3}, R_{planet} 1821 Km and σ_y of 100 Mpa. We find that the predicted height of mountains on Io is 54.4 Km.

Io has prodigious amounts of volcanic activity, but also possesses non-volcanic mountains that appear to be tilted crustal blocks. In reality, these mountains top out at only ~17km. So something else is limiting their height.

Io's average heat flux is a whopping 2.5 W/m^2 (Earth's is a comparatively measly 0.08 W/m^2), but most of that comes though local areas of volcanic activity. In general, only a few percent (let's say about 0.1 W/m^2) is conducted through the lithosphere. When rocks get to about half their melting temperature then they stop being able to support elastic stresses for long periods.

With this info, and the above diagram, in mind, how high can mountains on Io get? (Thermal conductivity is about 3 Wm/K, rock melts at ~1200K and Io's surface temperature is ~100K.)

The above figure tells us that most of the elastic stress is supported at a depth of 0.865 times the width of the mountain (assumed to be close to the height of the mountain here). Bigger mountains have stress supported at greater depths, but if these deep rocks are too warm (> $T_m/2$, where T_m is the melting temperature) then they stop behaving elastically and can't support the mountain.

With the conductivity (k) and heat flux (Q) supplied above we can estimate when the temperature reaches 600K ($T_m/2$). Heat flux is given by: $Q = k\left(T_z - T_{surface}\right)/z$
Rearranging for z: $z = k\left(T_z - T_{surface}\right)/Q$ and substituting in values from the question then z = 15km.

If this maximum depth of elastic behavior is roughly equal to the depth at which the highest mountains (height h) are supported then: z =0.865*h or h = 17.3km. Not that far off!

If Io's mantle has a density of 3300 Kg m^{-3} then how deep of a crustal root would be required to support a 17km high mountain through Airy Isostasy? Knowing what you now know about Io's internal temperatures is this a reasonable way to support these mountains?

If mountains on Io are floating because they have crustal roots that displace mantle material then we can balance the weight of the mountain with the buoyancy force on the root per unit area.

$$\rho_c gh = (\rho_m - \rho_c) gh_r$$

$$h_r = \left(\frac{\rho_c}{\rho_m - \rho_c}\right) h$$

If h is 17km and the crustal and mantle densities are 3000 and 3300 kg m^{-3} respectively then the depth of the root must be 170km (in addition to whatever the average crustal thickness is).

Airy Isostasy is therefore very inefficient on Io because the crust/mantle density contrast is small. We've seen in the last question that rocks reach half their melting temperature on Io at depths of only ~15km. Extrapolating this further downwards would imply reaching the melting temperature at depths of only about 33km. So maintaining a coherent crust root to depths > 170km would be impossible.

In summary, we looked at a few options for supporting Io's 17km high mountains. Support by strength alone is limited by the very thin elastic lithosphere on Io. So even though gravity there is low, mountains cannot reach their full potential height. Similarly support by Isostasy seems unlikely, as distinct crustal roots are unlikely to survive to the great depths needed to support these mountains.
Support by material strength where the depth of support is limited to be within the elastic lithosphere provides the best explanation.

4) **If roughly 10 major basins (>900 km in diameter) formed on the Moon during late heavy bombardment. How many craters greater than 1km in size formed during this period?**

Three separate power laws describe the cumulative number of craters above a certain diameter. Each applies over a separate diameter range:

$N_{cum}=C_3\ D^{-2.2}$ for 64 km $<$ D
$N_{cum}=C_2\ D^{-1.8}$ for 2 km $<$ D $<$ 64km
$N_{cum}=C_1\ D^{-3.8}$ for D $<$ 2 km

Adding up the number of craters in each size range.

$N(>64km) = C_3\ (64km)^{-2.2}$
$N(>2km)-N(>64km) = C_2\ (2km)^{-1.8} - C_2\ (64km)^{-1.8}$
$N(>1km)-N(>2km) = C_1\ (1km)^{-3.8} - C_1\ (2km)D^{-3.8}$

Since these three power laws intersect at 2km and 64km then many of these terms cancel. The total is: $C_1\ (1km)^{-3.8}$, or just C_1 so long as we work in kilometers.

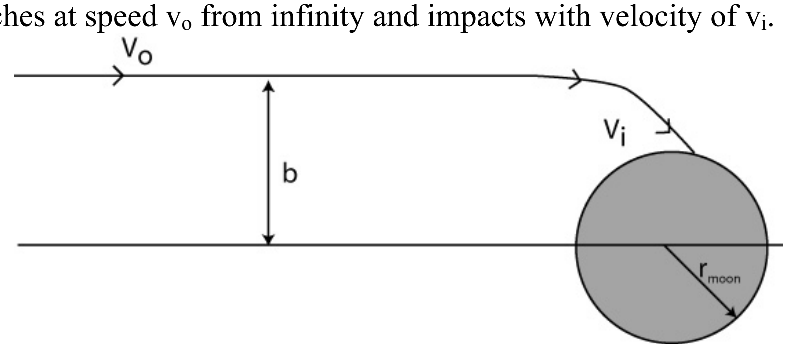
At D of 64km $C_3\ (64km)^{-2.2}= C_2\ (64km)^{-1.8}$
So $C_2 = C_3\ (64km)^{-0.4}$
At D of 2km $C_2\ (2km)^{-1.8}= C_1\ (2km)^{-3.8}$
So $C_1 = C_2\ (2km)^2 = C_3\ (64km)^{-0.4}(2km)^2=0.758\ C_3$

For the largest basins, $10= C_3\ (900\ km)^{-2.2}$
So $C_3 = 3.15743 \times 10^7$ (keeping the distance units as kilometers).
So $C_1 = 2.393 \times 10^7$

The 10 large basins imply that there are 23.9 million craters greater than 1km.

Derive the gravitationally enhanced cross-section of the Moon:

The body approaches at speed v_o from infinity and impacts with velocity of v_i.



We need to conserve both energy and angular momentum during this process.
Energy at distance = energy upon impact:

$$\tfrac{1}{2}mv_o^2 = \tfrac{1}{2}mv_i^2 - \frac{GMm}{r_{moon}}$$

$$v_o^2 = v_i^2 - \frac{2GM}{r_{moon}} = v_i^2 - v_{esc}^2$$

Angular momentum at distance = angular momentum upon impact:

$$m\,b\,v_o = m\,r_{moon}\,v_i$$

$$v_i = \frac{b}{r_{moon}}v_o$$

Substituting into the energy conservation equation for v_i gives:

$$v_o^2 = \left(\frac{b}{r_{moon}}v_o\right)^2 - v_{esc}^2$$

$$v_o^2\left(1 - \left[\frac{b}{r_{moon}}\right]^2\right) = -v_{esc}^2$$

$$\left[\frac{b}{r_{moon}}\right]^2 = 1 + \frac{v_{esc}^2}{v_o^2}$$

$$\pi b^2 = \pi r_{moon}^2\left(1 + \frac{v_{esc}^2}{v_o^2}\right)$$

How many hit the Earth during the same late heavy bombardment period? What speeds do they hit at? How much extra impact energy did the Earth receive compared to the Moon?

The Earth and Moon interact with the same projectile population. We can use the number of lunar impacts derived above and the ratio of gravitation cross-sections to find the number of objects that hit the Earth.

$$R = \frac{\pi b_{EARTH}^2}{\pi b_{moon}^2} = \frac{\pi r_{EARTH}^2\left(1 + \frac{v_{esc\_earth}^2}{v_o^2}\right)}{\pi r_{moon}^2\left(1 + \frac{v_{esc\_moon}^2}{v_o^2}\right)}$$

$$R = \frac{r_{EARTH}^2\left(v_o^2 + v_{esc\_earth}^2\right)}{r_{moon}^2\left(v_o^2 + v_{esc\_moon}^2\right)}$$

Earth's escape velocity is 11.19 kms^{-1}, the Moon's escape velocity is 2.38 km s^{-1}.
Earth's radius is 6371km, the Moon's radius is 1738km.
So, R = 20.4. The moon experienced 23.9 million impacts so the Earth experienced about 488 million impacts.

Impact energy scales as velocity squared. From the conservation of angular momentum above we see that:

$$v_{i\_moon} = \frac{b}{r_{moon}} v_o$$

$$so: \left(\frac{v_{i\_earth}}{v_{i\_moon}}\right)^2 = \left(\frac{b_{earth}}{b_{moon}}\right)^2 \left(\frac{r_{moon}}{r_{earth}}\right)^2 = R\left(\frac{r_{moon}}{r_{earth}}\right)^2 = 1.52$$

We can assume that the projectiles hitting the Earth and the Moon have the same mass distribution. The difference in energy is therefore a factor of 1.52. The difference in the number of impacts was a factor of 20.4, so in total the Earth receives about 31 times more impact energy.

The actual impact speeds are given by one of the equations above: $v_i^2 = v_o^2 + v_{esc}^2$
Escape velocity for the Moon and the Earth are 2.38kms^{-1} and 11.2kms^{-1} respectively, so the impact speeds are 15.19 kms^{-1} and 18.72 kms^{-1}.

> **When the velocity is constant like this, does most of the delivered impact energy come from the rarer large impacts or the more numerous small ones (and does the same hold true for craters less than 1km in size)?**

The energy delivered by crater in a certain size range depends on the number of craters and how much energy each one takes to form. The number of craters between diameter D and $\sqrt{2}$ D is given by:

$$N\left(D \to \sqrt{2}D\right) = cD^{-b} - c\left(\sqrt{2}D\right)^{-b}$$

$$N\left(D \to \sqrt{2}D\right) = c\left(1 - 2^{-b/2}\right)D^{-b}$$

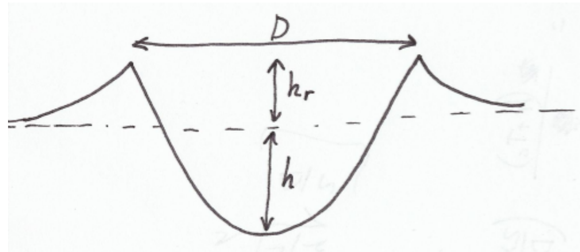
$$N\left(D \to \sqrt{2}D\right) \propto D^{-b}$$

Energy needed to form a crater size D is proportional to D^3 according to Lampson's law.
So total energy deposited by craters size D$\rightarrow\sqrt{2}$ D is proportional to D^{3-b}.
If the exponent (3-b) is greater than 0, then larger D's mean larger amounts of energy.
If the exponent (3-b) is less than 0, then larger D's mean smaller amounts of energy.

b is 1.8 or 2.2 for craters 2-64km and >64km respectively. So (3-b) is above zero and larger impacts deliver more energy in those diameter ranges.
b is 3.8 for craters <2km so (3-b) is less that zero and smaller impacts deliver more energy in this diameter range.

OPTIONAL EXTRA CREDIT QUESTION BELOW

5) *Crater shapes.* Simple craters tend to be parabolas with h/D ~ 0.2. Ejecta blankets decrease in thickness according to the distance from the crater center cubed. If volume is conserved in the crater creation process then derive the height of the rim (h_r) relative to the depth of the crater (h).



First, find the volume missing below ground level (V_b). We know the crater bowl is a parabola and that the height is $h+h_r$ when the radius is D/2. So:

$$y = (h + h_r)\left(\frac{2x}{D}\right)^2 \qquad \text{valid for } x = 0 \text{ to } D/2$$

when y=h (ground level) then $$x_o = \frac{D}{2}\sqrt{\frac{h}{(h + h_r)}}$$

The volume found in the usual way by adding up concentric cylinders of radius x, height y and thickness dx.

$$V_b = \int_0^{x_o} 2\pi\, x\, y\, dx$$

$$V_b = \frac{8\pi}{D^2}(h + h_r)\left|\tfrac{1}{4}x^4\right|_0^{x_o}$$

$$V_b = \frac{2\pi}{D^2}(h + h_r)\frac{D^4}{16}\frac{h^2}{(h + h_r)^2}$$

$$V_b = \frac{\pi D^2 h}{8}\frac{h}{(h + h_r)}$$

The volume above ground level (V_a) is slightly trickier in that you need to integrate the volume under the remaining part of the parabola ($x=x_o$ to $x=D/2$) and the volume beyond the rim separately. Beyond the rim, the thickness of the ejecta blanket falls off with distance from the crater center cubed i.e.

$$y_{ejecta} = h_r\left(\frac{2x}{D}\right)^{-3} \qquad \text{valid for } x = D/2 \text{ to } \infty$$

Again doing the integral (notice that the height used in the first part is [y-h] rather than y) gives:

$$V_a = \int_{x_o}^{D/2} 2\pi\, x\, (y-h)\, dx \quad + \quad \int_{D/2}^{\infty} 2\pi\, x\, y_{ejecta}\, dx$$

$$V_a = 2\pi \int_{x_o}^{D/2} \left((h+h_r)\left(\frac{4}{D^2}\right)x^3 - hx \right) dx \quad + \quad \frac{2\pi D^3 h_r}{8} \int_{D/2}^{\infty} x^{-2}\ \ dx$$

$$V_a = 2\pi \left| (h+h_r)\frac{x^4}{D^2} - \frac{h}{2}x^2 \right|_{x_o}^{D/2} \quad + \quad \frac{2\pi D^3 h_r}{8}\frac{2}{D}$$

$$V_a = 2\pi \left((h+h_r)\frac{D^2}{16} - 2h\frac{D^2}{16} - \frac{D^2}{16}\frac{h^2}{(h+h_r)} + \frac{D^2}{16}\frac{2h^2}{(h+h_r)} \right) \quad + \quad \frac{\pi D^2 h}{8}\left(4\frac{h_r}{h}\right)$$

$$V_a = \frac{\pi D^2 h}{8} \left(\left(1+\frac{h_r}{h}\right) - 2 - \frac{h}{(h+h_r)} + \frac{2h}{(h+h_r)} \right) \quad + \quad \frac{\pi D^2 h}{8}\left(4\frac{h_r}{h}\right)$$

$$V_a = \frac{\pi D^2 h}{8} \left(5\frac{h_r}{h} - 1 + \frac{h}{(h+h_r)} \right)$$

If volume is conserved then $V_a = V_b$:

$$\frac{\pi D^2 h}{8} \left(5\frac{h_r}{h} - 1 + \frac{h}{(h+h_r)} \right) \quad = \quad \frac{\pi D^2 h}{8}\frac{h}{(h+h_r)}$$

$$5\frac{h_r}{h} - 1 = 0$$

$$\frac{h_r}{h} = \frac{1}{5}$$

The rim height is 1/5 of the crater depth (or 1/6 of the total relief).