

**PTYS 411 – Geology and Geophysics of the Solar System
Solutions for Homework #3**

1) *Heating from Al²⁶*. The following table appeared in McCord and Sotin (2005). C₀ is the mass fraction of that element in the silicate and H₀ is the heat produced by 1 kilogram of the element (not the rock) at time zero.

Table 5. Radiogenic Heating Parameters Used in the Models^a

Element	C ₀ , ppb	H ₀ (mW/kg of Element)	Half-Life, Myr
²⁶ Al	450	138	0.716
⁶⁰ Fe	0.8	74.7	1.5
⁴⁰ K	430	0.0619	1250
²³² Th	130	0.0204	14000
²³⁵ U	17.5	0.401	704
²³⁸ U	52.4	0.104	4470

Show that the heat produced at any one moment in time is $\frac{\lambda H C_0}{\mu} e^{-\lambda t}$ per kilogram of rock.

Where H is the energy per decay, μ is the atomic weight and λ is the decay constant.

In 1kg of rock there is C kg of the radioactive element (or C₀ kg at time zero). The number of particles this corresponds to is N=C/ μ (or N₀=C₀/ μ at time zero) where μ is the atomic mass of the element in question. We know the number of decays is proportional to the number of particles and that λ is the constant. So:

$$\frac{\partial N}{\partial t} = -\lambda N$$

Which has the usual solution of:

$$N = N_0 e^{-\lambda t}$$

$$\text{or } C = C_0 e^{-\lambda t}$$

So the number of decays at any point in time is given by: $\lambda N = \frac{\lambda C}{\mu} = \frac{\lambda C_0}{\mu} e^{-\lambda t}$

If each decay releases energy H, then the energy released (per kg) at time t: $H \frac{\lambda C_0}{\mu} e^{-\lambda t}$

Show also that $H_0 = \frac{\lambda H}{\mu}$.

H₀ is the heat produced by 1 kilogram of the element (not the rock) at time zero. In this case we need to find the heat produced at t=0 and C₀ = 1. Sticking those values into the just-derived

expression gives: $H_0 = \frac{\lambda H}{\mu}$.

Incidentally, we can use this to simplify the original expression for heat production. Heat production is given by:

$$H \frac{\lambda C_0}{\mu} e^{-\lambda t} = H_0 C_0 e^{-\lambda t}$$

Aluminium 26 dominates at the earliest times, how much heat is each element producing per kilogram at t=0? Use values in the above table.

From the formula immediately above, heat production in 1kg of rock at t=0 is $H_0 C_0$
Calculations are shown in the table below. ^{26}Al is clearly the most energetic, producing more than 100 times the energy of the nearest competitor ^{60}Fe .

Element	C_0 , ppb	H_0 , mW/kg	H @ t_0 , pW/kg
^{26}Al	450	138	62100
^{60}Fe	0.8	74.7	59.76
^{40}K	430	0.0619	26.617
^{232}Th	130	0.0204	2.652
^{235}U	17.5	0.401	7.0175
^{238}U	52.4	0.104	5.4496

How long before Fe^{60} is providing more heat than Al^{26} , how long before the long-lived isotopes e.g. U^{235} are more important than ^{26}Al ?

The heat production rate of an element is given by $H_0 C_0 e^{-\lambda t}$

If two elements (A & B) produce the same amount of energy at time t then:

$$H_{0-A} C_{0-A} e^{-\lambda_A t} = H_{0-B} C_{0-B} e^{-\lambda_B t}$$

$$t = \frac{\ln\left(\frac{H_{0-A} C_{0-A}}{H_{0-B} C_{0-B}}\right)}{(\lambda_A - \lambda_B)} = \frac{\ln\left(\frac{H_{0-A} C_{0-A}}{H_{0-B} C_{0-B}}\right)}{\ln(2)} \left(\frac{1}{T_{1/2-A}} - \frac{1}{T_{1/2-B}} \right)^{-1}$$

Using values appropriate to ^{26}Al for element A and values appropriate to ^{60}Fe for element B we find this time to be 13.72 Myr.

Using values appropriate to ^{26}Al for element A and values appropriate to ^{235}U for element B we find this time to be 9.4 Myr

The earliest solids (CAI's in chondrites) had initial concentrations of Al^{26} of 6×10^{-5} times that of Al^{27} . That ratio is only 8×10^{-6} in the earliest chondrites, how much time elapsed before these chondrites formed.

^{27}Al doesn't change with time so in this case the number of ^{26}Al atoms is $6 \times 10^{-5} N_{\text{AL27}}$ at time zero and the number of atoms fell to $8 \times 10^{-6} N_{\text{AL27}}$ at some later time. The ratio of these numbers is given by $\exp(-\lambda t)$ and is conveniently independent of N_{AL27} . Solving, we find that time is given by $\ln(6 \times 10^{-5} / 8 \times 10^{-6}) / (\lambda) = [\ln(6 \times 10^{-5} / 8 \times 10^{-6}) / \ln(2)] * T_{\text{half}}$ which has a value of 2.08 Myr.

2) **Bingham flows.** The driving stress in a Bingham fluid must exceed a finite yield value in order for flow to start. Equate the stresses at the base of the flow to this yield value to get an expression for the thickness of the flow. For a given slope and lava composition, are lava flows thicker on Earth or Moon?

Let's say that the flow has a thickness H , density ρ , and is on a slope α . If the yield stress is σ_y then:

$$\sigma_y = \rho g H \sin \alpha$$

$$H = \frac{\sigma_y}{\rho g \sin \alpha}$$

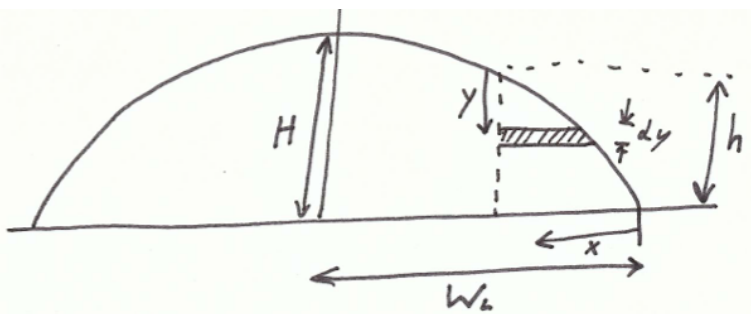
Note that σ_y is not a frictional force that depends on normal stress. The material moves by internal deformation (flow), it doesn't slide down the slope. There's a minor inconsistency here that often gets glossed over because slopes are usually low.

The sine in the above expression should be a tangent if H is measured normal to the slope.

Since thickness scales inversely with gravity, flows will be thicker in a lower gravity environment (like that of the lunar surface).

The flow spreads out laterally until the driving stresses fall below this yield stress. Since the pressure driving this lateral spreading increases with depth show that the flow will take on a parabolic cross-section.

The outward force on all the horizontal slices within the flow is balanced by the resistance to shear at the base. As the flow spreads out as much as it can, this shear stress falls to the yield stress at which point the flow stops spreading.



$$\int_h^0 \rho g y \, dy = -\sigma_y \, x$$

$$\frac{1}{2} \rho g h^2 = \sigma_y \, x$$

$$h = \sqrt{\left(\frac{2\sigma_y}{\rho g} \right) x}$$

i.e. flow thickness (h) varies as the square root of distance from the edge (a parabola). The width of the levee therefore depends on the thickness of the flow (H).

$$W_L = \frac{\rho g}{2\sigma_y} H^2$$

Show that the minimum width of the flow is $\frac{\sigma_y}{\rho g \sin^2(\alpha)}$ (where σ_y is the yield stress and α is the slope). For a given slope and lava composition, are lava flows wider on Earth or the Moon?

How is the flow width to thickness ratio related to the slope? Will this be the same on Earth and the Moon?

From the previous parts of the question we know that $W_L = \frac{\rho g}{2\sigma_y} H^2$ and that

$H = \frac{\sigma_y}{\rho g \sin \alpha}$. The flow width W_f , is at least $2W_L$ (the flow may also have a flat central portion). Combining these relations gives:

$$w_f = \frac{\rho g}{\sigma_y} H^2$$

$$w_f = \frac{\rho g}{\sigma_y} \left(\frac{\sigma_y}{\rho g \sin \alpha} \right)^2$$

$$w_f = \frac{\sigma_y}{\rho g \sin^2 \alpha}$$

Flow width scales inversely with gravity, so flows will be wider in a lower gravity environment (like that of the lunar surface).

We can also use the above relationships to eliminate the yield stress to figure out another relation between the flow's slope, thickness, and width.

$$w_f = \frac{H}{\sin \alpha}$$

$$\frac{H}{w_f} = \sin \alpha$$

i.e. flow thickness to width ratio scales with the sine of the slope. This is independent of gravity

(Adapted from Melosh 2011)

Steep flow fronts are observed at the edges of broad, extensive lava flows on the lunar Mare. These lava flows are considerably thicker than terrestrial lava flows, often reaching 100 m in height. The average slope of one such flow is about 0.5°. Compute the Bingham yield stress of this lava flow. Typical terrestrial basaltic lava flows have yield stresses of several thousand Pa. Is lunar magma substantially stronger than terrestrial magma?

We've seen that $H = \frac{\sigma_y}{\rho g \sin \alpha}$. Putting in numbers for lunar gravity (1.62 ms^{-2}),

H(100m), slope (0.5°) and density (3000 kgm^{-3}) we find that the yield stress is: 4240 Pa. i.e. not all that dissimilar from terrestrial basalt.

3) Show crater number density in a equilibrium population is: $N_{ceq} = c_{eq} D^{-2}$

The surface is saturated with craters when each subsequent impact removes an old one. The area of a crater is: $0.25\pi D^2$. If N is the maximum number of craters per unit area you can have then:

$$N\left(\frac{\pi}{4}D^2\right) = p_f$$

Where p_f is the packing efficiency (maximum of 0.905 for hexagonal packing). This is the geometric saturation level and never occurs in reality. A surface is saturated when the number of craters reaches some fraction (E_f) of this value.

$$N = E_f p_f \left(\frac{\pi}{4}D^2\right)^{-1} = \left[4/\pi p_f E_f\right] D^{-2}$$

$$N = c_{eq} D^{-2} \quad \text{where} \quad c_{eq} = \left[4/\pi p_f E_f\right]$$

In our case E_f is 4% so c_{eq} is 0.046. i.e. independent of age or crater size.

Find an expression for D_{eq} in terms of c_{eq} and c.

Saturation exists for craters less than a certain diameter (D_{eq}). At this diameter the crater population can be described by both the saturation and production power laws.

$$N = c_{eq} D_{eq}^{-2} = c D_{eq}^{-b}$$

$$\text{so: } D_{eq}^{2-b} = \left(\frac{c_{eq}}{c}\right) \quad \text{or} \quad D_{eq} = \left(\frac{c}{c_{eq}}\right)^{1/b-2}$$

The whole premise of dating a surface by crater counting is that the number of craters increases linearly with time (constant impact rate). In the power law that describes a crater population on a surface, c is proportional to time, and since c_{eq} is just a constant then c/c_{eq} is also proportional to time. So:

$$D_{eq} = (\text{time})^{1/b-2}$$

Show the fractional area of craters $D \rightarrow D_{eq}$ is:

$$f(D, D_{eq}) = \frac{\pi b c_{eq}}{4(b-2)} \left[\left(\frac{D_{eq}}{D}\right)^{b-2} - 1 \right]$$

The number of craters between diameter D and $D+\Delta D$ per unit area is given by:

$$N(D \rightarrow D + \Delta D) = cD^{-b} - c(D + \Delta D)^{-b}$$

$$N(D \rightarrow D + \Delta D) = cD^{-b} \left(1 - \left(1 + \frac{\Delta D}{D} \right)^{-b} \right)$$

$$\text{Assume: } \frac{\Delta D}{D} \ll 1$$

$$N(D \rightarrow D + \Delta D) = cbD^{-b} \frac{\Delta D}{D} = cbD^{-b-1} \Delta D$$

The area of each crater is $0.25\pi D^2$ so the area from craters between diameter D and $D+\Delta D$ is given by:

$$\Delta \text{area} = \frac{\pi}{4} D^2 cbD^{-b-1} \Delta D = \frac{\pi cb}{4} D^{1-b} \Delta D$$

This is area per unit area so is the fraction of surface covered. Integrate this fractional coverage from crater diameter to D_{eq} gives the total fraction of the surface covered by craters in this diameter range:

$$f(D, D_{eq}) = \int_D^{D_{eq}} \frac{\pi cb}{4} D^{1-b} \Delta D = \frac{\pi cb}{4(2-b)} \left| D^{2-b} \right|_D^{D_{eq}}$$

$$f(D, D_{eq}) = \frac{\pi cb}{4(2-b)} D_{eq}^{2-b} \left[1 - \left(\frac{D}{D_{eq}} \right)^{2-b} \right]$$

Recalling from earlier in the question that: $D_{eq}^{2-b} = \left(\frac{c_{eq}}{c} \right)$

Making this substitution, changing signs in the denominator and within brackets and changing sign of the exponent:

$$f(D, D_{eq}) = \frac{\pi bc_{eq}}{4(b-2)} \left[\left(\frac{D_{eq}}{D} \right)^{b-2} - 1 \right]$$

What is the average regolith thickness?

If $f=1$ at $D=D_{avg}$ then:

$$1 = \frac{\pi b c_{eq}}{4(b-2)} \left[\left(\frac{D_{eq}}{D_{avg}} \right)^{b-2} - 1 \right]$$

$$\text{or rearranging: } \left(\frac{D_{eq}}{D_{avg}} \right) = \left[\frac{4(b-2)}{\pi b c_{eq}} + 1 \right]^{\frac{1}{b-2}}$$

Average regolith thickness is $\langle h \rangle$ and given by $D_{avg}/4$, so rearranging gives:

$$\langle h \rangle = \left(\frac{D_{eq}}{4} \right) \left[\frac{4(b-2)}{\pi b c_{eq}} + 1 \right]^{\frac{-1}{b-2}}$$

If $b = 3.8$ then

$$\langle h \rangle = 0.058 D_{eq}$$

If $D_{eq} = 250\text{m}$ then $\langle h \rangle = 14.4\text{m}$

We know that $\langle h \rangle \propto D_{eq}$ and $D_{eq} \propto \text{time}^{1/b-2}$

So: $\langle h \rangle \propto \text{time}^{1/b-2}$

How long does it take to build up a 1m regolith?

From the above proportionality we know: $\left(\frac{1\text{m}}{14.4\text{m}} \right) = \left(\frac{t}{3.8\text{Gyr}} \right)^{1/b-2}$

Solving for t gives 31 Myr.

How thick will the regolith be in another 5Gyr?

From the above proportionality we know: $\left(\frac{x}{14.4\text{m}} \right) = \left(\frac{8.8\text{Gyr}}{3.8\text{Gyr}} \right)^{1/b-2}$

Solving for x gives 23m.

- 6) **Satellite heating. If Io's surface heat flux is 3 W m^{-2} then how much heat is produced per kilogram in the interior? Compare this with what a typical piece of solar system rock (chondrite) produces via radioactive decay i.e. $4 \times 10^{-12} \text{ W kg}^{-1}$. What fraction of Io's heat comes from radioactivity rather than tides?**

Io's radius: 1821.3 km

Io's surface area: $4.17 \times 10^{13} \text{ m}^2$

Io's heat flux at the surface: 3 W m^{-2}

Multiply the heat flux by the total surface area to get the heat produced in the interior.

Io's total heat production: $1.25 \times 10^{14} \text{ W}$

Io's mass is $8.92 \times 10^{22} \text{ Kg}$

So heat produced per unit mass is: $1.4 \times 10^{-9} \text{ W Kg}^{-1}$

This is ~351 times what a chondrite produces via radioactive decay (we're ignoring the fact that Io has an iron core that doesn't produce radioactive heat in this comparison).

Tides must be supplying 99.7% of Io's internal heat production.

If Europa has liquid water 4km below the surface and the average surface temperature is 110K then what is Europa's heat flux? How much radiogenic heat is produced in the rocky portion of Europa via radioactive decay? What fraction of Europa's heat comes from radioactivity rather than tides?

Assuming the ocean starts at a temperature of 273K, we can relate depth to heat flux with the usual thermal conduction formula:

$$q = k \frac{(273 - T_s)}{z}$$

The thermal conductivity (k) of water ice varies strongly with temperature. The value appropriate to the surface temperature (110K) is $\sim 4 \text{ W m}^{-1} \text{ K}^{-1}$, whereas the value appropriate to ice near the freezing point ($\sim 273\text{K}$) is $\sim 2 \text{ W m}^{-1} \text{ K}^{-1}$. Here, we'll just use the average of $3 \text{ W m}^{-1} \text{ K}^{-1}$. Substituting in values of 4000m for z and 110K for T_s , we find that Q must be:

$$q = 122 \text{ mW m}^{-2} \quad \text{to explain Europa's ice shell thickness.}$$

Europa's water layer is mostly liquid and is ~150km thick. So:

Mass of Europa's water: $4.2 \times 10^{21} \text{ Kg}$

Mass of Europa: $4.8 \times 10^{22} \text{ Kg}$

So the mass of Europa rocky interior is: $4.38 \times 10^{22} \text{ Kg}$

Using the chondritic production rate (and again ignoring the fact there is an iron core) we see that Europa's radiogenic heat production is $1.75 \times 10^{11} \text{ W}$

When conducted out through Europa's surface area, this corresponds to 5.7 mW m^{-2}

The actual heat flux we estimated was 122 mW m^{-2} , so the balance (116.3 mW m^{-2}) must be due to tidal heating. Expressed as a fraction of the total this is:

$$\frac{122 \text{ mW m}^{-2} - 5.7 \text{ mW m}^{-2}}{122 \text{ mW m}^{-2}} = 95\%$$

So the percentage contribution from radiogenic/tidal heating is 5/95 %.

If there were no tidal heating on Europa the how thick would the ice-shell be?

If there were no tidal heating then the heat flux (q) would be 5.7 mW m⁻² rather than 122 mW m⁻². Rearranging the heat conduction formula and using previous values for k and T_s.

$$q = k \frac{(273 - T_s)}{z} \quad \text{or} \quad z = \frac{k}{q} (273 - T_s)$$

We find that z is 85789m. i.e. without tides the ice would be ~86km thick (still less than the 150km of water on Europa, so a liquid ocean would still exist).

How thick would the ice shell be on Ganymede if the rocky portion of that body produces radiogenic heat at the chondritic rate and the surface temperature is similar to Europa?

Here, we'll assume that tidal heating at Ganymede is negligible and calculate the heat flux produced by radioactive decay.

Radius of Ganymede's rocky core is 68% of the body: 0.68 * 2634km = 1791km

Density of the rocky interior is ~3400 Kg m⁻³

Mass of the rocky interior: 8.2 x 10²² Kg

Heat production (assuming chondritic rate): 3.28 x 10¹¹ W

(We've again ignored the iron core)

Surface area of the whole moon, including the H₂O exterior is: 8.72 x 10¹³ m²

Heat flux at the surface: 3.76 mW m⁻²

Using the same approach as in the last part of the question and using previous values for k and T_s, we find that this heat flux implies z is 130km.

Based on the magnetometer data the actual liquid layer is thought to be 100km deep. So what's the reason for the difference? We ignored several things like the real variation of k with temperature, the possibility of salts of ammonia lowering the melting temperature, convection within the ice and the effects of pressure on the melting point of the ice.