PTYS 411 - Geology and Geophysics of the Solar System
Solutions for homework #4

---

1) If the core volume decreases by a factor F, then show that the surface area of the planet decreases by a factor: $1-\frac{2(1-F)}{3}\left(\frac{R_c}{R_P}\right)^3$

Assume F is 0.995, how many square kilometers did Mercury loose?

The volume of the planet's core changes from V_{core} to FV_{core}, causing the surface area of the whole planet to change from S_p to XS_p, we would like to find X. As all the volume change is caused by the change in core size. The change in the planets volumes $\Delta V_p = (1\text{-}F)V_{core}$. We can relate volume and surface area of a sphere:

$$S_p = 4\pi R_p^2$$

$$so: \quad \frac{1}{3} S_p^{3/2} = \frac{1}{3}(4\pi)^{3/2} R_p^3$$

$$rearrange: \quad \frac{S_p^{3/2}}{3\sqrt{4\pi}} = \frac{4}{3}\pi R_p^3 = V_p$$

We'll make the 1st order approximation that: $\Delta V_p = \frac{\delta V_p}{\delta S_p}\Delta S_p$

$$\text{since}: \quad V_p = \frac{S_p^{3/2}}{3\sqrt{4\pi}} \quad then: \quad \frac{\delta V_p}{\delta S_p} = \frac{S_p^{1/2}}{2\sqrt{4\pi}} = \frac{\sqrt{4\pi R_p^2}}{2\sqrt{4\pi}} = \frac{R_p}{2}$$

$$\Rightarrow \Delta V_p = \frac{R_p}{2}\Delta S_p \quad or \quad \Delta S_p = \frac{2\,\Delta V_p}{R_p}$$

The change in surface area, ΔS_p, is $(1\text{-}X)S_p$ so:

$$X = 1 - \frac{\Delta S_p}{S_p} = 1 - \frac{2\ \Delta V_p}{S_p\, R_p} = 1 - \frac{2\ (1-F)V_{core}}{\left(4\pi R_p^2\right)R_p}$$

$$X = 1 - \frac{(1-F)\,\frac{4}{3}\pi R_{core}^3}{2\pi R_p^3}$$

$$X = 1 - \frac{2(1-F)}{3}\left(\frac{R_{core}}{R_P}\right)^3$$

If F=0.995, (and R_{core}/R_P = 0.75 for Mercury) then X=0.9986. So Mercury lost 0.14% of its surface area, which is ($R_{mercury}$ = 2440km) roughly 105,200 km^2.

Mercury's lithosphere broke along many thrust faults during this episode. If each fault is about 500km long and has a displacement of 2km, how many faults does Mercury need to accommodate this shrinkage?

If we assume a typical fault dip of about 45 degrees then a displacement of 2km causes ~1.4km of surface to be overridden by the thrust sheet. If the faults are 500km long then that corresponds to ~700 km^2 to be lost. As Mercury lost 105,200 km^2 in total that corresponds to about 150 faults.

This system of global thrust faults is unique to Mercury, yet the other terrestrial planets also possess cooling cores. Why don't we see this happen on the Earth, Venus or Mars? The answer is different for each body.

Earth: The surface is already split into plates that can slide under each other to accommodate any global shrinkage.

Venus: Venus was resurfaced ~700 Myr ago, there hasn't been much core cooling in the meantime so F (and by extension X) is small.

Moon: The lunar core is very small or perhaps non-existent. $(R_{core}/R_P)^3$ will be very small for the Moon and by extension so will X.

Mars: Mars has a very thick lithosphere so it is harder to generate the thrust faults. It has also been recently argued that the sulfur in the Martian core will ensure that it stays completely molten, even until today, so F is very small.

2) **Impacts on Venus: Show that the ram pressure equals:** $P_{ram} \approx v^2 \rho_{atmosphere}$

The projectile sweeps up atmospheric particles in its path and changes the momentum of those particles so that their velocity equals the impactor velocity. The force needed to change their momentum leads to a reactionary force on the impactor. The resulting pressure is the force per unit area and is known as the ram pressure.

The change in the velocity of the atmospheric particles is from $0 \rightarrow v$ ms^{-1}. The change in momentum is the mass of this material times the velocity change. A unit area (1 m^2) on the front of the impactor moves through the atmosphere at a velocity v, so the volume it sweeps out in one second is v x 1m^2, and the mass it sweeps out is $v\,\rho_{atmosphere}$. So the momentum change every second is: $v \times v\,\rho_{atmosphere} = v^2\,\rho_{atmosphere}$. This is the force per unit area, and so the pressure.

The pressure variation with height is given as: $P(z) \approx P_S\, e^{-z/H} \quad where \quad H = kT/g\mu_{ATM}$ **. Convert the atmospheric pressure equation to density. What is the atmospheric surface density and scale height for Venus, Earth and Mars. Use temperatures of 750, 270 and 200K and surface pressures of 100, 1, 0.01 bars respectively.**

Start from the ideal gas law: $PV = nRT$, where n is number of moles, R is the universal gas constant, V is volume and T is temperature. This can be rewritten in terms of the number of particles (N) rather than number of moles:

$$PV = NkT$$

where k is the Boltzmann constant. After some rearrangement:

$$P = \frac{N\mu_{ATM}}{V}\frac{kT}{\mu_{ATM}}$$

where μ_{ATM} is the mass of one molecule. So:

$$P = \rho\frac{kT}{\mu_{ATM}} \quad or \quad \rho = \left(\frac{\mu_{ATM}}{kT}\right)P$$

$$so: \quad \rho = \left(\frac{\mu_{ATM}P_s}{kT}\right)e^{-z/H} = \rho_S e^{-z/H}$$

The molecular mass for atmospheres on Mars and Venus is that of CO_2 and on Earth is that of N_2. Given values for the surface pressure (P_s) and atmospheric temperature, we can calculate the scale height (H) and surface atmospheric density (ρ_s) to be:

	Pressure(z=0)	**Atm. T**	$H = kT/g\mu_{ATM}$	$\rho_S = \left(\frac{\mu_{ATM}P_s}{kT}\right) = \frac{P_s}{gH}$
Mars	**0.01 bars**	**200 K**	**10.01 km**	**0.027 kg m^{-3}**
Earth	**1 bar**	**270 K**	**8.12 km**	**1.272 kg m^{-3}**
Venus	**100 bars**	**750 K**	**15.88 km**	**71.918 kg m^{-3}**

If an impactor barely makes it to the surface without fragmentation on Mars, at what altitude will it break up if it had hit Venus.

If the impactor falls from rest at infinity it will reach the surface of Mars at 5.03 kms^{-1}. The ram pressure at the surface given the surface atmospheric density (calculated above) times the impact velocity squared and is 6.83×10^5 Pa. If the body breaks up at his point (and not just because it slammed into the ground at 5 kms^{-1}) then this is the strength of the body.

If the same body again falls from rest but instead impacts Venus then the velocity will be 10.36 kms^{-1}. The ram pressure at the surface given the density calculated above and this velocity is given by:

$$P_{ram} = v^2 \rho_{ATM} = v^2 \rho_S e^{-z/H} = \left(7.72 \times 10^9 \, Pa\right) e^{-z/H}$$

When the body breaks up the ram pressure is equal to the strength we already calculated so:

$$6.83 \times 10^5 \, Pa = \left(7.72 \times 10^9 \, Pa\right) e^{-z/H}$$

$$z = -H \ln\left(\frac{6.83 \times 10^5 \, Pa}{7.72 \times 10^9 \, Pa}\right) = 9.33 \, H = 148.2 \, km$$

Alternatively, we could assume the same impact velocity for both planets so it would effectively cancel out of the expression for breakup-elevation. In this case, we need to find out what elevation the atmosphere on Venus has an equivalent density to that at the surface of Mars.

$$\rho_{S\_MARS} = \rho_{S\_VENUS} e^{-z/H}$$

$$z = H \ln\left(\frac{\rho_{S\_VENUS}}{\rho_{S\_MARS}}\right) = 7.89 \, H = 125.3 km$$

One way to recognize meteors is by their fusion crust i.e. the exterior if the rock is melted during its passage through the atmosphere. How hot do the gases at the leading edge of the meteor get, just before impact into the martian surface? (assume they're adiabatically compressed). Is this hot enough to melt rock? How deep does this thermal disturbance penetrate into the meteorite?

We can calculate the temperature of the atmospheric gasses at the leading edge of the meteor by assuming they have been adiabatically compressed to the ram pressure. For adiabatic compression of an ideal gas: $PV^\gamma = const.$, where gamma is the ratio of specific heats (~7/5) for air.

Combining this with the ideal gas law:

$$V = \frac{NkT}{P}$$

$$so: \frac{V_2}{V_1} = \frac{T_2 P_1}{T_1 P_2} \quad and \quad \left(\frac{V_2}{V_1}\right)^{\gamma} = \left(\frac{T_2 P_1}{T_1 P_2}\right)^{\gamma}$$

$$\frac{const}{const} = \frac{P_2 V_2^{\gamma}}{P_1 V_1^{\gamma}} = \left(\frac{T_2}{T_1}\right)^{\gamma} \left(\frac{P_2}{P_1}\right)^{1-\gamma}$$

$$T_1 = T_2 \left(\frac{P_2}{P_1}\right)^{\frac{1-\gamma}{\gamma}}$$

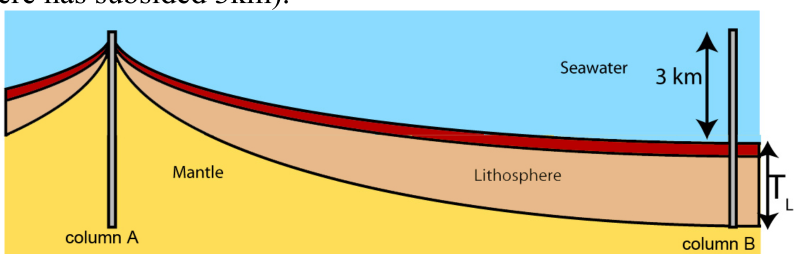
T_2 and P_2 are 200 K and 0.01 bars and P_1 is the ram pressure (calculated above as 6.83×10^5 Pa). Both the pressure and ram pressure fall off exponentially with the same scale height so the ratio of these surface values should apply to the rest of the atmosphere as well. Inputting these values gives a heated air temperature of 1260 K. Enough to barely melt the front face of the meteor; but, there isn't enough time to conduct much heat into the interior so this molten layer is thin.

The thermal disturbance is conducted to a depth $d = \sqrt{\kappa t} = \sqrt{\frac{k}{\rho c} t}$

Typically (for basalt) k = 2.5 W/m/K, c = 800 J/Kg/K and ρ=3000 kg/m^3.
The time is that taken to traverse the atmosphere, a few seconds.
So d ~ 2mm i.e. pretty thin.

3) *Oceanic lithosphere subduction.* **At some distance from the spreading ridge, the lithosphere is 100km thick and the seafloor has subsided by 3km. Assuming the uncooled mantle has a density of 3300 Kg m^{-3} and that the plate is isostatically supported everywhere, is this plate ready to be subducted?**

The figure below shows a cartoon of this situation (with scale very distorted). Isostatic equilibrium implies that the total weight of vertical columns should be the same everywhere. Since everything above the spreading ridge is seawater and everything below the lowest portion of the lithosphere is mantle material we need only consider material between these two vertical limits when adding up the weight of column A (at the spreading center) and column B (at some distance away where the lithosphere has subsided 3km).



The weight of the column A material: $\rho_m g(3000m + T_L)$

The weight of the column B material: $\rho_S g T_L + \rho_w g(3000m)$

Where ρ_m, ρ_S *and* ρ_w are the densities of the mantle, slab and water respectively.

Equating these gives:

$$\rho_S g T_L + \rho_w g(3000m) = \rho_m g(3000m + T_L)$$

$$\rho_S = \frac{\rho_m(3000m + T_L) - \rho_w(3000m)}{T_L}$$

Using the values provided, we find ρ_S to be 3369 kg m^{-3}, which is slightly denser than the underlying mantle material. So yes, the slab can be subducted.

What's the density ratio between the cooled mantle material now part of the lithosphere and the uncooled mantle material? Assume a 5km thick crust with a density of 3000 kg m^{-3}.

The slab density is 3369 kg m^{-3}, which is made up of 5km of crust at 3000 kg m^{-3} and 95km of cooled mantle material (together forming the 100km thick lithosphere). The weighted average can be written as:

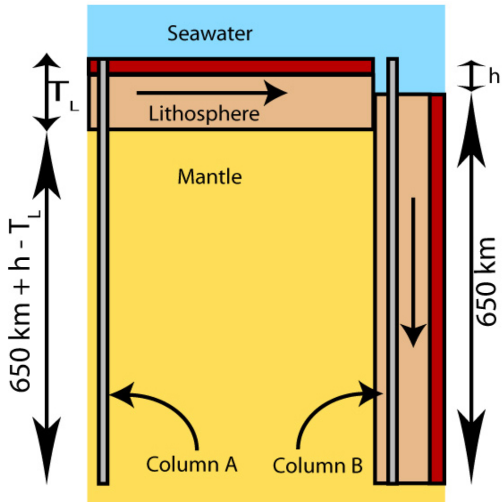
$$\rho_S = \frac{5}{100}\rho_c + \frac{95}{100}\rho_{cool\ mantle}$$

The density of the cooled mantle material is therefore:

$$\rho_{cool\ mantle} = \frac{100\rho_S - 5\rho_c}{95} = 3388\ kg\ m^{-3}$$

The density ratio between the cooled and uncooled mantle material is 1.027.

At the Mariana trench the lithosphere basically takes a right angle turn and plunges vertically into the mantle at least 650 km. Right above the subducting slab the Marianas Trench has formed. Calculate how deep you'd expect this trench to be.



As before we assume isostatic equilibrium and add up mass in two equivalent columns. Column A is far from the trench and column B runs through the trench and subducting lithospheric slab. The arrangement is shown in the cartoon on the left.

The weight of the column A material: $\rho_m g(650km + h - T_L) + \rho_s g(T_L)$

The weight of the column B material: $\rho_S g(650km) + \rho_w gh$

Where ρ_m, ρ_S *and* ρ_w are the densities of the mantle, slab and water respectively. Equating these and rearranging gives:

$$\rho_m g(650km + h - T_L) + \rho_s g(T_L) = \rho_S g(650km) + \rho_w gh$$
$$h(\rho_m - \rho_w) = \rho_S(650km) - \rho_m(650km - T_L) - \rho_s(T_L)$$
$$h = \frac{(\rho_S - \rho_m)(650km - T_L)}{(\rho_m - \rho_w)} = 16.5km$$

The trench is predicted to be 16.5km deep.

The actual trench-bottom is about 3.5km below the surrounding seafloor what's the reason for the discrepancy between your answer and this number?

This prediction far exceeds the observed trench depth, the reason for this is that the density of the mantle increases with depth so there is in actuality a larger buoyancy force which acts to reduce the trench depth.

4) Sand dunes on Triton? (From Chap 9 of Melosh 2011)
Triton, Neptune's largest moon, possesses a very thin atmosphere that is composed mainly of N_2 gas at a chilly 38 K. Nevertheless, geysers spout plumes 8 km high into the atmosphere. Suppose that loose "sand" grains of ice (perhaps from impact ejecta) lie on the surface. How fast do the winds of Triton have to blow to just entrain such ice grains? Compute both the minimum friction velocity needed to loft these grains and the minimum wind speed 1 meter above the surface. Compare this velocity to the speed of sound in Triton's atmosphere. What can you conclude about the probability of finding "sand" dunes on Triton when it is be visited by a spacecraft with an imaging system capable of resolving such features? Facts that you may find useful: The viscosity of nitrogen gas at 38 K is about 2.2×10^{-6} Pa-s and its density at Triton's atmospheric pressure of 1.5 Pa is 1.3×10^{-4} kg/m^3. The acceleration of gravity at the surface of Triton is 0.78 m/sec^2.

The adjusted weight of a particle is: $F_{down} = \frac{\pi}{6} d^3 (\rho_s - \rho_a) g$

The upward force is proportional to the drag force: $F_{drag} = \frac{\pi}{4} d^2 \ \frac{1}{2} C_D \rho_a u_*^2$
Equating these at the motion threshold with a constant of proportionality:

$$u_{*T} = A\sqrt{\left(\frac{\rho_s - \rho_a}{\rho_a}\right) gd}$$

Experiments show that A ~0.1 for a fully turbulent flow. Using sand sized grains of water ice (d=200 microns, ρ_s=920 kg/m^3) on Triton gives: u_{*T} = 3.3 ms^{-1}
The speed 1m above the surface can be found from the 'law of the wall': $\frac{u}{u_*} = \frac{1}{\kappa} \ln\left(\frac{z}{z_o}\right)$

The parameter z_o is usually d/30 for closely packed grains and κ (Von Karman's constant) is ~0.4. So, for a sandy bed on Triton, when z=1m then u is 98.3 ms^{-1}.

Sound speed in an ideal gas is given by: $c = \sqrt{\gamma \frac{p}{\rho}}$

Where γ is the ratio of specific heats and is 7/5 for a diatomic gas. Plugging in the numbers reveals than c on Triton is 127ms^{-1}. So the wind (at z=1m) would need to be blowing at a substantial fraction of the speed of sound to mobilize sandy material (incidentally, such winds would be blowing at mach-1 32m above the surface). This is an unlikely situation so, sadly, we will probably never see dunes on Triton unless it had a denser atmosphere in the past.