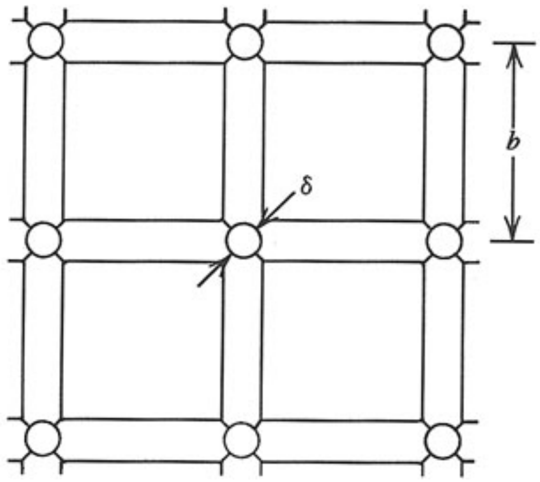
PTYS 411 – Geology and Geophysics of the Solar System
Solutions for Homework #5



1) Let's think of our permeable medium as a solid rock that's been fractured into approximately cubic fragments with cylindrical cracks along the cube edges. The cube centers are spaced a distance 'b' apart and the size of fractures (diameter of the cylinders) is δ. Show that the porosity of such a rock is:

$$\phi = \frac{3\pi}{4}\left(\frac{\delta}{b}\right)^2$$

The porosity is the void space divided by the total volume. Firstly we calculate the void space associated with each cubic fragment of rock.

Each crack (cylindrical pipe) has a volume of $b\pi\left(\frac{\delta}{2}\right)^2$ and each crack is bordered by 4 cubes so the crack volume/per cube is $\frac{1}{4}\times b\pi\left(\frac{\delta}{2}\right)^2$. Each cube has 12 edges and a crack runs along each edge so the total volume of the cracks associated with one cube is: $12\times\frac{1}{4}\times b\pi\left(\frac{\delta}{2}\right)^2 = \frac{3\pi}{4}b\delta^2$.

Each cube (along with the pipes that border it) has a volume of b^3. So the porosity (ratio of these two volumes): $\frac{1}{b^3}\frac{3\pi}{4}b\delta^2 = \frac{3\pi}{4}\left(\frac{\delta}{b}\right)^2$

When a fluid flows through a pipe its mean velocity is given by: $\langle u\rangle = -\frac{\delta^2}{32\eta}\frac{dp}{dx}$. Use Darcy's law to show that the permeability is given by: $k = \frac{\pi}{128}\frac{\delta^4}{b^2}$

Darcy's law gives an expression for flux (volume per second) of water per unit area. We can use the information given to calculate this flux and later set it equal to the expression given by Darcy's law.

Mean velocity in the crack is <u>, so the flux of water carried by the pipe is $\langle u\rangle\pi\left(\frac{\delta}{2}\right)^2$. The cracks are separated by a distance b, so the area associated with each crack is b^2. The flux per unit area is therefore: $\frac{\langle u\rangle\pi}{b^2}\left(\frac{\delta}{2}\right)^2$.

Darcy's law gives the flux per unit area as: $-\frac{k}{\eta}\frac{\delta P}{\delta x}$. Comparing these fluxes and substituting in the expression for <u> gives:

$$\frac{\langle u\rangle\pi}{b^2}\left(\frac{\delta}{2}\right)^2 = -\frac{k}{\eta}\frac{\delta P}{\delta x}$$

$$-\frac{\delta^2}{32\eta}\frac{dp}{dx}\frac{\pi}{b^2}\left(\frac{\delta}{2}\right)^2 = -\frac{k}{\eta}\frac{\delta P}{\delta x}$$

$$canceling \quad and \quad rearranging: \ \frac{\pi}{128}\frac{\delta^4}{b^2} = k$$

If the cracks are 1mm wide and crack junctions are spaced 10cm apart, what are the numerical values for porosity and permeability?

Under these conditions porosity is $2.35\text{x}10^{-4}$ and permeability is $2.45\text{x}10^{-12}$ m^2, which is 2.45 darcies (a darcy is 10^{-12} m^2).

The hydrologic conductivity is given by: $K = \frac{k\rho g}{\eta}$.

What is the hydrologic conductivity ($k\rho g/\eta$) if the fluid is water on Mars? What is it for methane on Titan? What do you think this says about the effectiveness of groundwater flow on Mars vs Titan?

Water on Mars is likely to be close to freezing, its viscosity is ~$1.75\text{x}10^{-3}$ Pa s. Using this value, the permeability calculated above, Martian gravity, and the density of water we find K=$5.2\text{x}10^{-6}$ ms^{-1}.

Methane on Titan has a viscosity of ~$2\text{x}10^{-4}$ Pa s. Using this value, the permeability calculated above, Titan's gravity (1.35 m s^{-2}), and the density of methane (450 kg m^{-3}) we find K=$7.4\text{x}10^{-6}$ ms^{-1}.

Hydraulic conductivity on Mars and Titan are very similar! Despite the different gravities, fluid densities and viscosities.

*2) **Freezing of a Martian Ocean.*** **At some point the ice thickness increases from z to z+Δz in a time Δt.**
How much heat is released by this phase change per unit area?
How much heat can be conducted through a unit area of the ice slab in this time?

The heat released per unit area by freezing a layer of ice Δz meters thick is: L ρ Δz joules. In this time you can conduct: $k\frac{(273-T_S)}{z}\Delta t$ joules per unit area through the ice.

Equate these two quantities to show that the ice thickness (at time t after freezing starts) is given by: $z=\sqrt{\frac{2k}{L\rho}(273-T_S)}\,t^{1/2}$

Equating these two quantities gives:

$$k\frac{(273-T_S)}{z}\Delta t = L\,\rho\,\Delta z$$

$$k\frac{(273-T_S)}{L\,\rho}\Delta t = z\,\Delta z$$

Integrate this equation, the boundary condition is zero thickness at t=0.

$$\int_0^t k\frac{(273-T_S)}{L\,\rho}\Delta t = \int_0^z z\,\Delta z$$

$$k\frac{(273-T_S)}{L\,\rho}t = \frac{1}{2}z^2$$

$$z=\sqrt{\frac{2k}{L\,\rho}(273-T_S)}\,t^{-1/2}$$

If the surface temperature is 240K, how many years does it take to freeze an ocean 1km thick? [L=3.2x10^5 J kg^{-1}, ρ=920 kg m^{-3}, k =2 W K^{-1} m]
Rearranging the above equation:

$$t=\frac{z^2}{\frac{2k}{L\,\rho}(273-T_S)}$$

Plugging the provided values t=2.23x10^{12}s or 71,945 terrestrial years.

*3) **Porosity and Martian groundwater:*** **Porosity is thought to decrease, from its surface value, exponentially with depth in a regolith Usually the pressure in the near surface is just the weight of the overlying rock ρgZ. In this case, show that the pressure is given by:** $P = \rho g\left[z + H\phi_{z=0}\left(e^{-z/H} - 1\right)\right]$**.**

The weight of a square meter slab of regolith of thickness Δz is ρg Δz (1-φ) where Φ is the porosity. Weight per square meter is the pressure, so to get the total pressure at depth; we need to add up all these thin slices of regolith via an integral.

$$P = \int_{z=0}^{Z} \rho g(1-\phi)\,\Delta z$$

$$P = \int_{z=0}^{Z} \rho g\left(1-\phi_o e^{-z/H}\right)\,\Delta z$$

$$P = \left|\rho g z + \rho g H\phi_o e^{-z/H}\right|_0^Z = \rho g Z + \rho g H\phi_o e^{-Z/H} - \rho g H\phi_o$$

$$P = \rho g\left(Z + H\phi_o e^{-Z/H} - H\phi_o\right)$$

$$P = \rho g\left(Z + H\phi_o\left(e^{-Z/H} - 1\right)\right)$$

If porosity decreases in such a way so that it is 1% of the surface value when the pressure reaches 100 MPa, show that $H = \dfrac{100MPa}{\rho g\left(\ln(100) - 0.99\phi_{z=0}\right)}$**.**

If porosity is 1% of the surface value then $\varphi=0.01\varphi_0$ so:

$$0.01 = e^{-Z/H} \qquad or \qquad H\ln(100) = Z$$

Taking the expression from the previous section and substituting in these two values we find:

$$P = \rho g\left(Z + H\phi_o\left(e^{-Z/H} - 1\right)\right)$$

$$100MPa = \rho g\left(H\ln(100) + H\phi_o(0.01 - 1)\right)$$

$$100MPa = H\ \rho g\left(\ln(100) - 0.99\phi_o\right)$$

$$H = \frac{100MPa}{\rho g\left(\ln(100) - 0.99\phi_o\right)}$$

If surface porosity is 35%, what is H on Mars?

If $\varphi_0=0.35$, $g=3.72\ ms^{-2}$ and density is that of basalt $\rho=3000\ kg\ m^{-3}$. Then we find H to be 2104m.

Show that the regolith can store (per square meter) a volume of water given by: $H\varphi_{z=0}$. If this water were on the surface how deep would it be?

The pore space within a square meter slab of regolith of thickness Δz is φ Δz where Φ is the porosity. As before, to get the total pore spaces at all depths; we need to add up all these thin slices of regolith via an integral.

$$Pore\ space = \int_{z=0}^{Z} \phi\ \Delta z = \int_{z=0}^{Z} \phi_o e^{-z/H}\ \Delta z = \left| -H\phi_o e^{-z/H} \right|_0^{\infty} = H\phi_o$$

In this case we continue the integral to infinite depth although in reality the vast majority of the pore space is within a few scale-depths of the surface. The total pore space is therefore 736.4 m^3, and since water has unit density this depth this volume of water would be is 736.4m.

The pore-space water freezes from the surface downwards as the planetary heat flux declines. The current martian heat flux is about 30mW m^{-2}, how deep is the ice-water interface today if the mean surface temperature is ~240K and the regolith conductivity is 2 W K^{-1} m.

The planetary heat flux determines the temperature increase with depth. Heat flux is driven by the temperature gradient:

$$q = -k\frac{(T_1 - T_S)}{Z_1 - 0}$$

where T_s is the surface (z=0) temperature and T_1 is the temperature at some depth ($z=Z_1$). Using values quoted for heat flow and conductivity the depth at which T_1 reaches 273 is:

$$Z_1 = -k\frac{(T_1 - T_S)}{q} = 2200m$$

4) River profiles: The shear stress at the base of a flow is given by: $\boldsymbol{\tau = \rho g h\, sin(slope)}$. Rivers adjust their beds (by eroding or depositing material) so that this shear stress tends to be held constant. If river depth increases linearly downstream then show that the elevation of the river bed is given by:

$$\boldsymbol{z = c\, ln\left(\frac{L}{x}\right)}$$

where c is a constant and L is the distance between the drainage divide and the ocean.

$$\frac{\tau}{\rho g} = h\, sin(slope) \approx -h\frac{dz}{dx}$$

Depth increases linearly with distance downstream (h= γx, where γ is some constant) so:

$$\frac{\tau}{\rho g \gamma} \approx -x\frac{dz}{dx}$$

If everything on the left hand size is being held constant (c=$\frac{\tau}{\rho g \gamma}$) then:

$$\frac{c}{x}dx \approx -dz$$

$$c\, ln(x) \approx -z + k$$

k is the integration constant. Z=0 (sea level) when x=L, so: $k = c\, ln(L)$

$$c\, ln(x) \approx -z + c\, ln(L)$$

$$c\, ln\left({}^{L}/_{x}\right) \approx z$$

Let's say that c=50m and L=100km in one example.
If the river depth is 0.5m, 25km from the ocean then what is the shear stress on the river bed?

The shear stress at the base of a flow is given by: $\tau = \rho g h\, sin(slope)$.
We know the depth (0.5m) and gravity and density. We approximate the sin(slope) by -dz/dx:

$$z = c\, ln\left(\frac{L}{x}\right)$$

so: $\frac{dz}{dx} = c\left(-\frac{L}{x^2}\right)\left(\frac{x}{L}\right) = -\frac{c}{x}$

c is 50m and x is 75km (25km form the ocean) so sin(slope) (=-dz/dx) is 0.00067. Combining this with the expression for shear stress gives τ = 3.3 Pa for water on the Earth. On Titan (where gravity is 1.35 ms^{-2} and the density of liquid methane is 450 kg m^{-3}), the shear stress is lower, ~0.2 Pa.

What size particle (assume the sediment is quartz) do you expect to find on the bed at that location? What would these numbers be for the same river on Titan (i.e. liquid methane with water ice sediment)?

The shear stress needed to entrain a grain of diameter d is: $\tau = \theta(\rho_s - \rho_w)gd$
Assuming a fully-turbulent flow, θ is ~0.04.

For quartz sediment with a density of 2700 kg m^{-3} on Earth, 3.3Pa of shear stress can just about entrain grains of a diameter 5mm.

For water-ice sediment with a density of 920 kg m^{-3} on Titan, 0.2 Pa of shear stress can just about entrain grains of a diameter 8 mm.

The river's ability to move sediment decreases with downstream distance. Larger grains have been deposited on the bed further upstream and finer grains continue to be transported downstream. At this point in the river the 5mm (or 8mm on Titan) grains are at the threshold of motion so they accumulate here and make up most of the bed particles.

Bottom Line: Titan's rivers are very much like the Earth's !!

5) ***Martian Methane*. The global average concentration of methane on Mars was measures at 10 ppb by volume (number of particles). How many methane molecules are in the Martian atmosphere?**

A unit-area column of atmosphere has a mass P_s/g, where P_s is the surface pressure. So multiplying this by the area of the planet gives the mass of the entire atmosphere.

$$4\pi R_{Mars}^2 \frac{P_S}{g}$$

Dividing by the mass of a CO_2 molecule (the dominant component of the atmosphere gives the number of molecules. Multiplying this number of molecules by the volume mixing ratio of methane gives the number of methane molecules as:

$$4\pi R_{Mars}^2 \frac{P_S}{g} \frac{10\times 10^{-9}}{\mu_{CO_2}}$$

Substituting in values for martian gravity, atmospheric pressure and planetary radius gives the number of methane molecules as: $3.27\text{x}10^{33}$

If a molecule has a life-time of $2\text{x}10^{10}$s before being photodissociated then what is the methane production rate? (In Kg per year)

All these molecules will decay within the next $2\text{x}10^{10}$ seconds so the number of molecules decaying per second is $1.64\text{x}10^{23}$. Multiplying this by the molecular weight of methane (16 amu) gives a mass rate of: $4.38\text{x}10^{-3}$ kg s^{-1} or 135,718 kg yr^{-1}.

A cow produces an incredible 600 liters of CH_4 a day! Convert this to kg of methane per year and compare to the Martian production rate…. Based on this analysis, how many cattle are needed on Mars to maintain the planets methane concentrations?

600 liters a day is 219,150 liters a year. At standard temperature and pressure this corresponds to 9783.5 moles. For methane, this corresponds to 156.5 kg.

Martian methane needs to be resupplied at a rate of 135,718 kg yr^{-1} to maintain current levels. This implies that a herd of 867 cattle would be able to do this. Of course, more likely explanations exist.