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A NOTE ON THE DESCRIPTION OF SURFACE ROUGHNESS USING FRACTAL DIMENSION

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Abstract. Self-affine fractals are useful models of the surfaces of rock fractures. The scaling properties of these surfaces are described by two parameters, the fractal dimension and the crossover length. Two methods for estimating the fractal dimension of a profile of a rough surface are compared, the divider method and the spectral method. It is shown that the two methods yield the same results, if the horizontal resolution at which the profile is measured is smaller than the crossover length. However, for resolutions greater than the crossover length, the divider method always gives a fractal dimension close to 1. To guide future work, the crossover length is estimated for typical joint surfaces and for the San Andreas fault. Additionally, a simple method is proposed to obtain the correct fractal dimension without prior knowledge of the crossover length.

Introduction

Fractures of all sizes, ranging from microcracks to joints and faults, are well known for their effects on the mechanical and transport properties of rock. Mechanical properties, such as bulk elastic constants and shear strength, are strongly affected by the presence of fractures [e.g. Goodman, 1976; Barton and Choubey, 1977; Walsh and Grosenbaugh, 1979; Brown and Scholz, 1986]. Fractures also control the hydraulic conductivity of crystalline and tight sedimentary rock [e.g. Gangi, 1978; Kranz et al., 1979; Walsh and Brace, 1984; Brown, 1987]. These effects arise from the fact that the surfaces composing a fracture are rough and mismatched at some scale. The shape, size, and number of contacts between the surfaces control the mechanical properties. The separation between the surfaces or the "aperture" determines the transport properties.

Walsh and Grosenbaugh [1979] showed that the normal stiffness of a fracture should vary approximately as the inverse of the rms asperity height. For interlocked surfaces, relationships between the rms surface slope and the peak shear strength of a joint have been suggested by Tse and Cruden [1979] following the experimental work of Barton and Choubey [1977]. Therefore, if the topography varies with surface size, so must the mechanical and transport properties. This remark must be tempered by the fact that the mechanical and transport properties of fractures depend not only on the topography of the individual surfaces, but also on how well the two surfaces are correlated [Brown et al., 1986]. However, study of the scaling properties of individual surfaces provides the groundwork for understanding the scaling properties of fractures.

Scaling, Fractals, and Crossover Length

The scaling properties of natural rock surfaces have been studied in detail [Brown and Scholz, 1985; Scholz and Aviles, 1986; Power et al., 1987]. In these studies, the surface heights were digitized at equally spaced intervals along a line to produce a profile. These results show that all natural rock surfaces, including bedding planes, tension cracks (joints), and frictional wear surfaces (faults), are remarkably similar. To a first approximation, all surfaces have power spectral density functions $G(k)$ of the form:

$$G(k) = Ck^{-\alpha} \quad (1)$$

where $k = 2\pi/\lambda$ is the wavenumber and λ is the wavelength or distance along the profile. The proportionality constant C varies among surfaces. The power α usually falls in the range $2 < \alpha < 3$. Sayles and Thomas [1978] found similar behavior for numerous other random surfaces.

Mandelbrot [1983] has suggested that fractals are useful mathematical models of rough surfaces, and indeed this has some physical basis [Termonia and Meakin, 1986]. In the present context, a fractal is a particular mathematical model of irregular geometry, wherein the scaling properties are described by the fractal dimension D. The fractal dimension can range between the topologic and the Euclidian dimensions. For example, a profile of a rough surface is topologically a line (dimension 1), but is defined in Euclidian 2-space, and the fractal dimension falls between 1 and 2. In one sense, the fractal dimension is a measure of how much space a particular function fills.

Two classes of fractals are distinguished, self-similar and self-affine. One familiar example of a self-similar fractal is Brownian motion in the plane of a microscope slide [e.g. Mandelbrot, 1983, plate 13]. If one traces on graph paper the path of the particle through time at two different magnifications, the two drawings will look statistically the same (i.e. have the same statistical moments). Since a simple change in magnification left the complexity of the curve unchanged, this process is self-similar. In this case the fractal dimension is D=2. However, if one defines a coordinate system on the microscope slide and graphs the x-position of the particle as function of time, then a self-affine fractal of dimension D=1.5 is obtained. To be precise, if we refer to this function as $B(t)$ and the time axis is rescaled by two positive numbers h and g, then the functions $h^{1-D}B(ht)$ and $g^{1-D}B(gt)$ are statistically the same, whereas $B(ht)$ and $B(gt)$ are not. In order to obtain statistically equivalent graphs, the position axis must be scaled differently than the time axis. Thus $B(t)$ is merely self-affine. As we will see, the distinction between self-similar and self-affine fractals has some bearing on the method used to estimate the fractal dimension.

Self-affine fractals can be used as models of rough surfaces since, when they are in the form of linear profiles, they have power spectral density functions of the form (1). When $2 \leq \alpha < 3$, the fractal dimension D can be estimated from the relation $D = 2.5 - \alpha/2$, corresponding to $1 < D \leq 1.5$ [Mandelbrot, 1983]. However, the above expression relating α and D may not be reliable when D>1.5 [Brown and Scholz, 1985]. Figures 1 and

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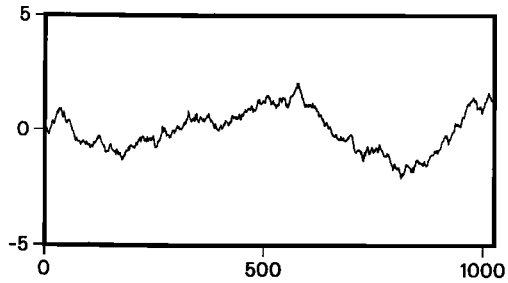

Fig. 1. Self-affine fractal with fractal dimension D=1.5, generated using the algorithm of Fournier et al. [1982].

2 give an example of a self-affine fractal and its power spectral density function.

The moments of the power spectrum provide a useful description of the surface roughness. The nth moment is defined as:

$$m_n = \int_{k_0}^{\infty} k^n G(k) dk \qquad (2)$$

where k_0 corresponds to the profile length λ_0. In practice the upper limit of integration is the Nyquist cutoff corresponding to a wavelength of twice the sample interval. The derivative theorem for the Fourier transform implies that m_0 is the variance (mean square value) of heights on the profile, m_2 is the variance of slopes, and m_4 is the variance of curvatures. When $G(k)$ represents a self-affine fractal then (2) gives $m_0 = \sigma^2 = \kappa {\lambda_0}^{2(2-D)}$, where κ depends on the constant C in (1). This is the scaling law for the rms height σ. For the case D=1.5, Sayles and Thomas [1978] refer to κ as the "topothesy," and they tabulate its value for various surfaces. Wong [1987] defines a different constant, the crossover length b, such that the standard deviation of heights is $\sigma = b(\lambda_0/b)^{2-D}$, thus $\kappa = b^{2D-2}$. The crossover length is in fact the same generalized topothesy suggested by Berry and Hannay [1978] in a comment to Sayles and Thomas [1978]. One important property of the crossover length is that when $\lambda_0 = b$ then $\sigma = b$.

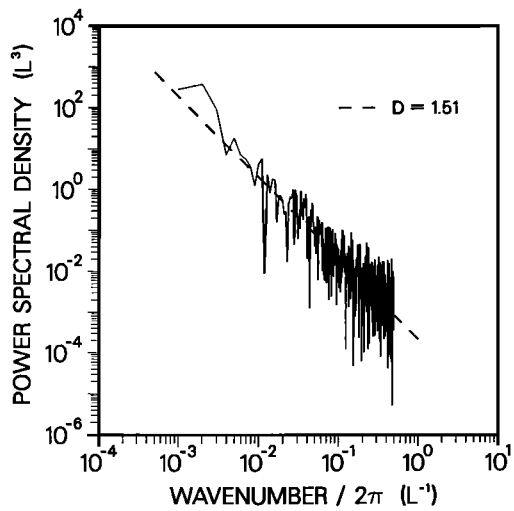

Fig. 2. Power spectral density function for the self-affine fractal of Figure 1. The dashed curve is a least squares fit for wavenumber/$2\pi < 3.0 \times 10^{-1}$. The slope gives the fractal dimension D=1.51.

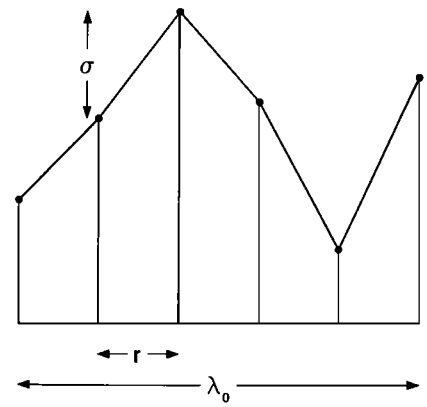

Fig. 3. Definition of terms used in the evaluation of the divider method for estimating the fractal dimension of a self-affine fractal.

Estimating the Fractal Dimension

The calculation of the fractal dimension from its precise definition is difficult, therefore less general alternative methods must be used in practice [Mandelbrot, 1983, 1985]. Each alternative method has some limits to its applicability. Two methods of estimating the dimension of a self-affine fractal have been used frequently. The spectral method has already been discussed. The other method is known alternatively as the ruler, compass, or divider method. This is performed conceptually by opening a pair of dividers to some distance r and walking them along the profile to estimate its total length. The total number of steps (total length of the line) is plotted as a function of r on a log-log plot. If linear, this curve has a slope of 1-D [Mandelbrot, 1983]. This method, or a derivation of it, will always give the correct value of D for self-similar fractals. This method has also been widely applied to self-affine fractals [e.g. Aviles et al., 1987; Okubo and Aki, 1987; Carr and Warriner, 1987; Turk et al., 1987]. However, the divider method will give the correct fractal dimension for self-affine fractals only under certain conditions [Mandelbrot, 1985; Wong, 1987], and indeed some values re-

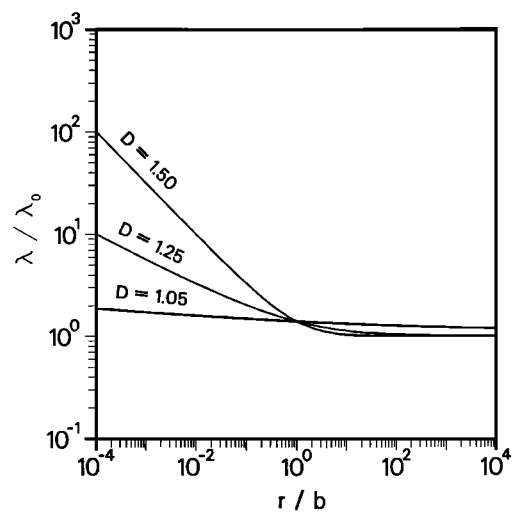

Fig. 4. Graph of (4) for several values of the fractal dimension, D. For $r \ll b$ the slope of the curves is $1 - D$, but for $r \gg b$ the slope is always close to zero.

ported from this method seem to be anomalously low. Thus, use of the divider method warrants some caution. To illustrate this point, I will recount an evaluation of the divider method given previously by Wong [1987] and follow with an example.

Suppose we have a self-affine fractal trace with a nominal length λ_0 digitized at discrete intervals of length r (Figure 3). The vertical fluctuations over the distance r are, on average, equal to the standard deviation of heights σ. There are λ_0/r segments in all and the total length of the line λ as a function of r is approximately:

$$\lambda = \frac{\lambda_0}{r}\left[r^2+\sigma^2\right]^{1/2} \quad (3)$$

Recalling that for a self-affine fractal over distance r the standard deviation of heights is $\sigma = b(r/b)^{2-D}$ [Wong, 1987], then (3) becomes:

$$\lambda = \lambda_0\left[1+(r/b)^{2(1-D)}\right]^{1/2} \quad (4)$$

The behavior of this equation is shown in Figure 4. Since the fractal dimension D > 1, then for $r \ll b$ we obtain $\lambda \approx \lambda_0(r/b)^{1-D}$. Thus log($\lambda$) vs. log($r$) has slope 1-D. However, for $r \gg b$ then $\lambda \approx \lambda_0$. In this case, calculation of D using the slope of the r-λ curve will always give D≈1. The crossover length b is interpreted as the horizontal resolution above which the divider method breaks down. Apparently, to obtain meaningful results from the divider method, one must have data digitized at a scale much smaller than the crossover length. However, even without prior knowledge of b a simple solution to this problem exists.

From the definition of b and (4) we notice that, if the ordinate of the self-affine fractal is multiplied by a constant greater than 1 so as to increase the standard deviation of heights σ, then the effective crossover length can be increased relative to the sample interval. Thus for a given range of divider lengths r, the limit $r \ll b$ can be reached and the correct fractal dimension obtained without actually changing the sample interval. A computer program was written to demonstrate this procedure, and the results of calculations for the self-affine fractal of Figure 1 are shown in Figure 5. The crossover length needn't be known ahead of time. One can simply magnify the profile height repeatedly by various factors until a stable estimate of D is obtained. In fact, whenever the divider method is used, the stability of the results should be tested in this manner.

The arguments just made were based on a modified version of the divider method, where the sample interval r was held constant at each step and the opening of the dividers varied. However, the same conclusions hold for the true divider method, where the divider lengths are held constant and the sample interval varies. While the results aren't shown here, a computer program implementing the true divider method was written and run for the self-affine fractal of Figure 1. The results are nearly identical to Figure 5.

The crossover length can be estimated from the power spectral density function. From (1), (2), and the definition of b the constant of proportionality in (1) is found to be $C = (4-2D)(2\pi)^{4-2D}b^{2D-2}$. However, plots of the power spectral density are normally presented in terms of the reciprocal wavelength $(1/\lambda)$ rather than the wavenumber k. In this case (1) can be written:

$$G(1/\lambda) = (4-2D)b^{2D-2}(1/\lambda)^{-(5-2D)} \quad (5)$$

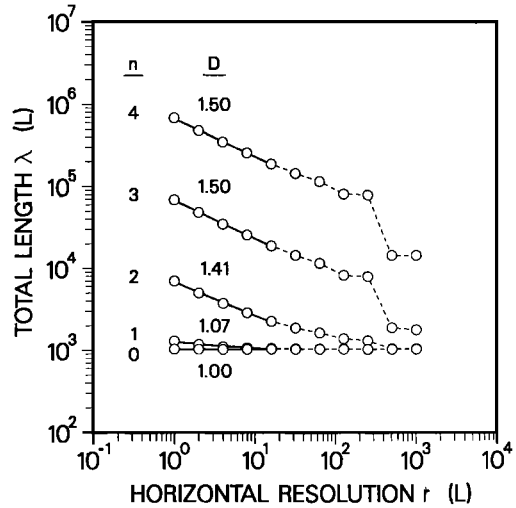

Fig. 5. The divider method applied to the self-affine fractal of Figure 1. The labels "n" represent the power of 10 (10^n) by which the vertical axis of the fractal was magnified prior to the application of the method. The labels "D" are the resulting fractal dimensions obtained from a least squares fit to the left-most five points on each curve.

This provides a technique to estimate b. From Figure 2 the crossover length is $b \approx 10^{-3}$. The rms roughness σ is proportional to $b^{1/2}$ when D=1.5. The maximum value of r used in Figure 5 to determine D is 16. Therefore to make r small, say $r < 0.1b$, the vertical axis of Figure 1 must be magnified by at least 400 ($= [10 \cdot 16 \cdot 1000]^{1/2}$). After this magnification, the divider method will give the correct fractal dimension (Figure 5). Using published plots of $G(1/\lambda)$ the crossover length is found to be $b \approx 10^{-8}-10^{-5}$ m for natural joints [Brown and Scholz, 1985] and $b \approx 5-100$ m for the 1906 section of the San Andreas fault [Scholz and Aviles, 1986].

Conclusions

Self-affine fractals are useful models of the surfaces of rock fractures. The scaling properties of these surfaces are described by two parameters, the fractal dimension and the crossover length. The divider method and the spectral method for estimating the fractal dimension of the profile of a rough surface are compared. Both methods yield the same results if the horizontal resolution at which the profile is measured is smaller than the crossover length, or equivalently if the amplitude of the profile is magnified appropriately. For resolutions greater than the crossover length, the divider method always yields fractal dimensions close to 1. This demonstrates that care must be taken when using the divider method to estimate the fractal dimension of a self-affine fractal.

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