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A Review of Methods Used to Determine the Fractal Dimension of Linear Features[1]

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An in-depth review of the more commonly applied methods used in the determination of the fractal dimension of one-dimensional curves is presented. Many often conflicting opinions about the different methods have been collected and are contrasted with each other. In addition, several little known but potentially useful techniques are also reviewed. General recommendations which should be considered whenever applying any method are made.

KEY WORDS: fractal measurement, divider method, box counting, spectral techniques, variogram.

INTRODUCTION

Although an increasing number of papers provide a theoretical basis for observing fractal behavior in geomorphological phenomena, the selection of a method which can provide a consistent and reliable determination of the fractal dimension remains unresolved. Many methods have been developed, but most have their practical and/or theoretical limitations. In this paper I will review many of the methods used to determine the fractal dimension of curves or one-dimensional profiles. Theoretical bases for the methods described below can be found in a number of texts and papers (e.g., Feder, 1988; Turcotte, 1992; Peitgen, Jürgens, and Saupe, 1992).

Determining the fractal dimension of linear features—natural features with a topological dimension of one—is a process undertaken in a wide variety of fields (Table 1). Linear features exist in their own right (e.g., coastlines, river networks, fault traces), and a wide variety of appropriate methods to determine their fractal dimension have been developed (Table 2). However, cuts or slices of greater dimensional phenomena (e.g., topographic contours or profiles) can also be analyzed using those same methods, methods which can be easier to

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Table 1. A selection of Applications of Methods Used to Determine the Fractal Dimensions of Linear Features

Method	Applied to	Comments	Citation
Area/perimeter	Digitized shorelines and contours	D ~ elevation (increased)	Goodchild, 1982
		D ~ constant from 0.5 km to 20 km	Kent and Wong, 1982
	Digital cloud images	D constant between 1 and 1000 km	Lovejoy, 1982
	Digital images of craters on Mars		Woronow, 1981
	Sinkhole perimeters	Larger sinkholes ~ fractal	Reams, 1992
Box counting	Digitized shorelines and contours	D ~ elevation (increased)	Goodchild, 1982
	Photographs of vegetation	Breaks in plot delimited vegetation types	Morse *et al.*, 1985
	River networks extracted from a DEM	Above a certain scale the network is space-filing (D ~ 2)	Tarbonton *et al.*, 1988
	Fracture patterns determined from remote imagery	D ~ multifractal from one area; D ~ 1.5 from another	Vignes-Adler *et al.*, 1991
Divider relation	Used physical dividers to measure lava flows	Lava shape is scale-invariant	Bruno *et al.*, 1992
	Digitized contours	D ~ elevation (increased)	Culling and Datko, 1987
	Digitized shorelines and contours	D ~ elevation (increased)	Goodchild, 1982
	Digitized shorelines	Definite break at a consistent distance	Kent and Wong, 1982
	Line skeletons of cave passages contours	Exhibited fractal behavior	Laverty, 1987
		D ~ elevation (increased)	Roy *et al.*, 1987
	Drainage basin perimeters	D ~ fairly consistent (between 1.06–1.12)	Breyer and Snow, 1992
	Digitized cartographic lines	D ~ scale of map	Müller, 1986, 1987
Kocak's law	Areas of lakes		Kent and Wong, 1982
Power spectrum	Natural rock surfaces	D ~ scale of analysis	Brown and Scholz, 1985
	Digital model of seafloor	D ~ scale and direction, from 1 to 100? km	Fox and Hayes, 1985
	Natural rock surfaces	D ~ scale and direction, from 10^{-5} to 10^{0} m	Power *et al.*, 1987
	Digitized traces of faults	D ~ scale from 10^{-5}?? to 10^{8} m	Scholz and Aviles, 1986

Table 1. Continued

Method	Applied to	Comments	Citation
Variogram	Various geophysical phenomena		Burrough, 1981, 1984
	Soil profiles	Found very high values of D	Burrough, 1984
	Soil pH	Found very high values of D	Culling, 1986
	Digitized maps	Consistently identified two fractal regimes	Culling and Datko, 1987
	DEMs	Mixed results; D ~ direction and the method used to create the DEM	Klinkenberg and Goodchild, 1992
	Ice sheet height profiles derived from Landsat data	Clear fractal structure; D ~ ice thickness	Rees, 1992

implement, and require far less computational power, than those methods which must be used when analyzing the greater dimensional phenomena. In addition, reducing the dimensionality of, for example, a surface or volume, means that the nature of the fractal can be altered (Mandelbrot, 1989). Thus, although a topographic surface or a cloud are self-affine fractals (see discussion below), horizontal slices or contours through those phenomena exhibit self-similar behavior, and the fractal dimension of the slice or contour can be directly related to that of the surface (Orey, 1970). This dimensionality-reduction process doesn't always alter the fractal nature since profiles of self-affine surfaces remain self-affine.

Determining the fractal dimension of a self-similar feature is generally easier than determining the fractal dimension of a self-affine feature, as will be discussed in the following sections. Determining whether a one-dimensional feature is self-affine is also a simpler process. If one can interchange or rotate the axes of the coordinate system used to map the feature without producing any fundamental changes, then the feature has the minimum requirements for self-similarity. For example, exchanging the (x, y) values which define a contour line alters nothing essential—both sets of axes carry the same information. However, one cannot exchange the (x, y) values of a topographic profile—even though both axes may be measured in the same units—since the vertical axis integrates different information (i.e., gravity) than does the horizontal axis. Even more obviously, the trace of a particle through time has axes which represent different information (time and distance). More formally, with self-affine fractals the variation in one direction scales differently than the variation in another direction (Mandelbrot, 1985).

Table 2. Methods Used to Estimate the Fractal Dimension[a]

Name of method	Relation	Estimate of the fractal dimension
Area/perimeter relation	$A \propto P^{2/D}$ A = area P = perimeter	Plot log A against log P, slope is $2/D$
Box counting	$n \propto b^{-D}$ n = number of filled boxes, b = box size	Plot log n against log b, slope is $-D$
Divider relation	$L(\tau) \propto \tau^{1-D}$ $L(\tau)$ = length of trail τ = step size	Plot log $L(\tau)$ against log τ, slope is $1 - D$
Korcak's law (Empirical relation for islands)	$N_r(A > a) \propto a^{-(D/2)}$ $N_r(A > a)$ = number of areas above size a	Plot log $N_r(A > a)$ against log a, slope is $-(D/2)$
Line-scaling	$X \sim N^{V_X}$; $Y \sim N^{V_Y}$ $H' = \dfrac{V_Y}{V_X}$ X, Y = standard deviation of x-, y-coordinates, respectively (see text for details)	If $H' \approx 1$, $D = \dfrac{1}{V}$ otherwise $D = 2 - H'$
Power spectrum	$P(\omega) \propto \omega^{-(5-2D)}$ $P(\omega)$ = the power ω = the frequency	Plot log $P(\omega)$ against log ω, slope is $-(5 - 2D)$
Variogram	$\langle [(Z_p - Z_q)^2] \rangle \propto (d_{pq})^{(4-2D)}$ Z_p, Z_q = elevations at points p and q d_{pq} = distance between p and q	Plot log $\langle [\ldots] \rangle$ against log d_{pq}, slope is $(4 - 2D)$

[a] $\langle\ \rangle$ = statistical expectation. For references see sources listed in Table I and throughout the text.

The distinction between self-affine and self-similar profiles is an elusive one, however (Mandelbrot, 1985). Consider a fracture surface. One could argue that since the forces which created the fracture acted differently in different directions (i.e., the stress was anisotropic), the resultant profile is analogous to a topographic profile, and should therefore be considered as a self-affine profile (see Sakellariou *et al.*, 1991 for a further discussion of the self-affinity of rock surfaces). That the material properties themselves will have an obvious effect on the nature of the profile adds a further complication. Thus, for analytical purposes, whether a profile is studied as a self-affine curve or as a self-similar

curve may ultimately depend on the objectives of the research (e.g., Carr and Warriner, 1987).

The number of methods which can be used to determine the fractal dimension of a linear feature is relatively large (Table 2), and new methods can (and are) always being derived. The type of linear feature analyzed, whether self-similar or self-affine, must be carefully considered when selecting an appropriate method. Some methods can work with features of either type, while others should only be applied to self-similar features or, if they are applied to self-affine features, must be applied in a very considered manner. For some methods, if the curve is self-similar, then the fractal dimension is computed one way. If the curve is self-affine, then the fractal dimension is computed in another way. The interpretation of the dimensions produced by the various methods also requires careful consideration of several aspects, such as whether the differences between (theoretical) expectations are the result of the method (i.e., the practical implementation) or of the fractal model (Brown, 1987; Dubuc *et al.*, 1989a; Andrle, 1992).

Two main aspects of method-based problems can be identified. Is the method being used in a proper manner? This aspect is discussed in detail in the following sections. The second method-based problem relates to the usually statistical means by which the fractal dimension is calculated. If least-squares regression is used to determine the slope of a curve—and for most methods the fractal dimension is some function of slope–statistical considerations must be properly observed or the derived parameters may be suspect. Although the slope can be determined from a hand-drawn line (e.g., Mark and Aronson, 1984), the results can be unreliable. For that reason, and since confidence limits for the derived parameters are often also desired, a statistical procedure such as least squares is the preferred method (McBratney and Webster, 1986). This aspect is considered further in the following sections.

All of the methods described in this review (Table 2) produce a monofractal dimension. The literature on multidimensional fractals is a developing one (Lovejoy and Schertzer, 1986; Mandelbrot, 1988, 1989; Feder, 1988). Few of the methods presented herein have been (or even could be) extended to multifractal dimensional analyses, as has, for example, the box counting method by Lovejoy and Schertzer (1987). It is important to separate the observation of discrete fractal elements (cf. Andrle and Abrahams, 1989, 1990) from truly continuous multidimensionality since, in most cases, the question of whether a mono- or a multidimensional fractal is the more appropriate model has not been answered. The results of any analysis must be carefully scrutinized for evidence of multidimensionality. One way to check for such behavior is to plot the residuals from the best-fitting line of slope and check for evidence of nonlinearity (Andrle, 1992).

One drawback common to every method of obtaining the fractal dimension

is that discretization of the phenomenon being investigated will result in a measured fractal dimension that is different from its theoretical fractal dimension (i.e., the dimension which would be obtained if we were able to work in continuous space with its infinite detail). The coarser the discretization, the greater the expected difference. However, the direction that that difference will take is open to question (Gilbert, 1989, Table 1) found instances of both a decrease and an increase in the fractal dimension with greater decimation of the original data. Thus, to assume that discretization—achieved by removing finer details—will always result in a lesser fractal dimension—as some authors have assumed (e.g., Shelberg, Moellering, and Lam, 1982)—is inappropriate. Since the implementations of some methods are themselves discretizations of continuous expressions, that, in itself, may affect the derived fractal dimension (Dubuc *et al.*, 1989a).

A different type of problem, one which afflicts many of the methods discussed below, is that the dimension of the whole may not equal the dimension of the parts. Consider, for example, the case of horizontal cuts (e.g., contours) of a self-affine surface. For most surfaces, this will produce a suite of self-similar curves (Fig. 1). Matsushita, Ouchi, and Honda (1991) prove that the fractal dimension of the entire suite of curves ($\mathbf{D}_{e.s.}$) will generally be greater than the fractal dimension of a single curve ($\mathbf{D}_{s.c.}$) extracted from that suite ($\mathbf{D}_{s.c.}$

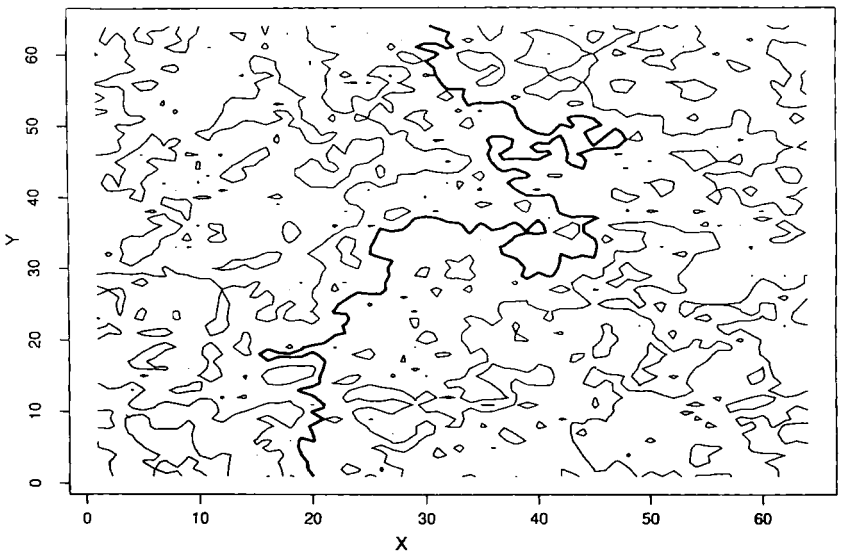

Fig. 1. Contour plot of a fractal surface (D = 2.35). The fractal dimension of the bolded contour line is calculated to be 1.21.

$\leq \mathbf{D}_{e.s.}$). The example they present, of curves derived from a fractal surface of dimension 2.5, shows $\mathbf{D}_{e.s.} = 1.50$ and, for the single longest curve, $\mathbf{D}_{s.c.} = 1.32$ (also see Ouchi and Matsushita, 1992).

Matsushita *et al.* (1991) present a formula which relates the fractal dimension of a single contour line to the fractal dimension of the surface: $D_{s.c.} = (2/1 + H)$. In comparing the results of that formula to the more often used $D = 2 - H$ we can see why so many studies have reported that contour-based dimensions are often lower than those reported for the surface (Fig. 2). This fact should be taken into consideration whenever a subset of the level set is used.

The divider method is arguably the most popular method of determining the fractal dimension of linear features (Gilbert, 1989; Clarke and Schweizer, 1991). It also has the most published variants, and it will be the first method considered in this paper. The second method which will be considered is the box counting method, another widely used method, particularly in physics and meteorology. This is, in part, because it can be applied to any dimensional set—a fact which increases its usefulness significantly. Following the review of the box counting method, there will be a discussion of the power spectrum method.

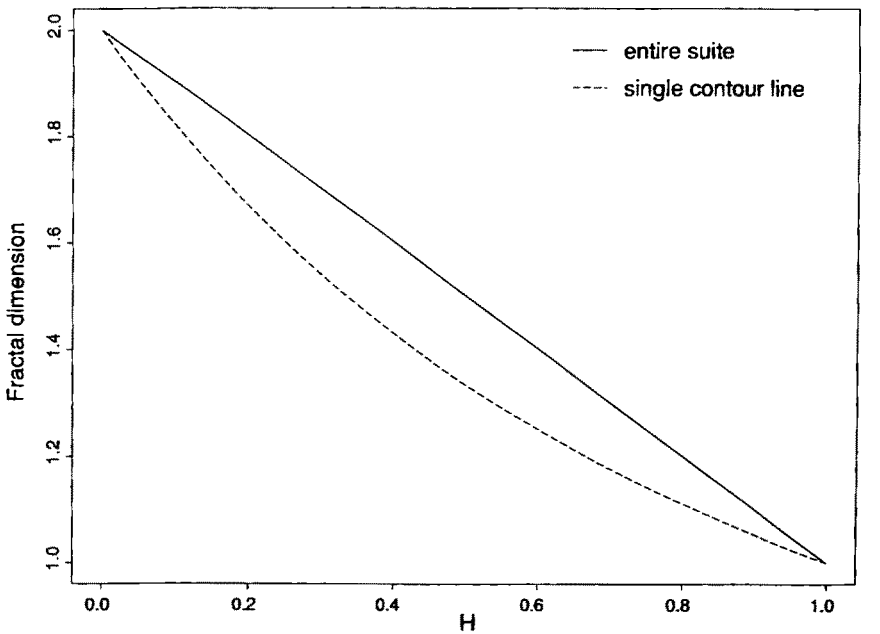

Fig. 2. The relation between the fractal dimension and the value of H, as determined by Matsushita *et al.* (1991).

This methodology requires the most sophisticated data preprocessing, a requirement which probably limits its applications. The variogram method and several area-scaling methods will then be discussed. Finally, the relative new line-scaling method of Matsushita and Ouchi (1989) will be presented, followed by a brief mention of several other methods.

Following the reviews of the various methods, some general conclusions will be made.

THE DIVIDER METHOD

The divider method has long been used to determine the length of cartographic lines (Maling, 1992). Richardson's (1961) investigations into the scale dependencies of border lengths, one of the key building blocks in the development of Mandelbrot's concept of fractional dimensions (Mandelbrot, 1967), has justifiably become one of the more cited references in the divider method literature. Because of the ease with which this method can be implemented—using either physical or computational dividers—a large number of studies have used the divider method to determine fractal dimensions of features ranging from particle shapes to lava flows (Table 1). This breadth of applications has resulted in a large number of independent reviews and a suite of contradicting recommendations (see below). As an illustration of the isolated efforts which can be observed, some have even referred to this method as being a "relatively new" technique (Power and Tullis, 1991)!

The divider method can be implemented in a number of ways, but the basic implementation is to "walk" the divider along the line and record the number of steps required to cover the line. By systematically increasing the width of the divider and repeating the stepping process, the relation between step size and line length over a range of resolutions can be determined. Calculation of the fractal dimension follows (Table 2).

The main problem with the divider method relates to the remainder—the fact that most often a non-integer number of steps is required to cover the line. The most commonly applied solution—the addition of the fractional step length—was found by Aviles *et al.* (1987) to produce slightly higher dimensions than when the number of steps was either rounded up or down. Mandelbrot (personal communication to anonymous reviewer) recommends retaining the remainder. While Gilbert (1989) concluded that the divider method was flawed and produced unreliable results, most authors have found the method to be fairly reliable when tested on curves of known fractal dimension (e.g., Aviles *et al.*, 1987; Peitgen, Jürgens, and Saupe, 1992). The results of practical applications have been somewhat mixed, however. In some cases, the divider dimensions have closely corresponded to those obtained by spectral analyses (Brown, 1987), variogram analyses (Roy, Gravel, and Gauthier, 1987), and other methods (Kent

and Wong, 1982). However, in other cases the divider dimensions have not corresponded at all (Carr and Benzer, 1991; Klinkenberg and Goodchild, 1992). Given the recent findings of Matsushita *et al.* (1991) it is not surprising that some differences have been reported.

Several different implementations of the divider method have been developed: Schwarz and Exner's (1980) fast algorithm, Batty and Longley's (1986) equipaced polygon method, Clark's (1986) HYBRID method, and Kennedy and Lin's (1986) FAENA method. These alternative implementations were developed primarily to speed up the "pacing" process. Given the ever increasing processing power of workstations, the need to look for computational shortcuts has been greatly reduced, and the traditional implementation should be the method of choice (cf. Hayward, Orford, and Whalley, 1989). In particular, since these implementations are all approximations of the original technique, they all produce dimensions different from those produced by the traditional implementation.

The divider method may also produce different dimensions because some methods—when determining the length of the curve—use the first point along the curve intersected by the divider as the next "walking" point (e.g., any of the fast methods, and the way that some authors appear to have implemented the divider method—Muller, 1986 is one example), whereas other methods use the last point intersected by the divider as the next "walking" point (e.g., the traditional approach). Mandelbrot (1986) demonstrates that the two choices produce different dimensions—$(1/H)$ and $2-H$, respectively. This explains why the divider dimension for an exact mathematical fractal such as a Koch curve is different from its theoretical dimension, since that dimension is determined, in effect, by using the first point intersected.

Additional problems with the divider method include a dependency of the fractal dimension on the starting position when counting, on the selection of the minimum and maximum step size, and the small islands problem (see below). As many authors have noted, *where* the divider commences its walk can greatly influence the results. To alleviate this problem, a large number of starting positions should be selected and the values averaged. Andrle (1992) suggests randomly selecting a point along the line and then stepping out to either end from that point. He found that after using approximately 50 randomly determined starting points, the standard deviation of the dimensions was minimized.

The selection of the smallest and largest step sizes is not without its problems also. Selecting too small an initial step size will result in length values that don't vary with step size. If such values are included in the slope determination, they will tend to decrease the slope and, thereby, reduce the apparent fractal dimension of the feature. Because of this, various criteria have been suggested for the selection of the smallest step size. The smallest step size should be: (1) twice the shortest distance between any two points (Andrle, 1992), and

(2) one-half the average distance between adjacent points (Shelberg, Moellering, and Lam, 1982). Selecting too large an initial step size will reduce the number of data points used in the slope determination.

Selecting the largest step size is a much more subjective decision. The larger the step size, the greater the effect that partial steps have and, as was discussed above, partial steps affect the value and variability of the derived dimension. Andrle (1992) recommended that the largest step size(s) not be used in order to minimize the effects of partial steps. Concomitantly, this reduces the likelihood that multiple fractal elements or multifractal behavior can be identified, since the fractal dimension would be calculated over a narrower range of scales.

Other reasons can be found for specifying a reduced maximum step size when using the divider method on self-affine curves. Wong (1987) and Brown (1987) demonstrated that the derived dimensions converge to a value of one if the step sizes are greater than a value referred to as the crossover length (b). If the maximum step size remains smaller than the crossover length, the dimension should be representative—what Mandelbrot (1985) refers to as the local fractal dimension (also see Peitgen and Saupe, 1988, p. 64 for further elaboration). When working with self-similar linear features the crossover length is not a concern.

A general rule for setting the maximum step size when working with self-affine features can be derived from the formula presented in Brown (1987): the greater the (expected) fractal dimension of the profile, the smaller the maximum step size should be relative to the standard deviation in heights. Since the crossover length is dependent on both the fractal dimension (which won't be known until after the measurement is made) and the vertical roughness (Brown, 1987; also see Mandelbrot, 1985, last sentence of section 7) suggests that greatly exaggerating the values of the vertical component significantly decreases the likelihood of the step size increasing past the crossover length. This is because the crossover length increases concomitantly. After an appropriate amount of magnification of the vertical component values, the derived fractal dimension should become stable (Brown and Scholz, 1985, Fig. 5).

The notion of a crossover length is not without its critics, however (Carr and Benzer, 1991). Rescaling the vertical component independently of the horizontal component changes the geometric relation between the two which, therefore, will change the results of the divider-length relation—an important distinction if one is interested in analyzing the apparent self-similarity of a profile. When scaling a self-affine curve, if the horizontal coordinates are multiplied by a factor r, the vertical coordinates should be multiplied by a factor of r^{2-D} (Mandelbrot, 1982; Malinverno, 1989). In light of this we must rationalize Brown's (1987) rescaling "fix." Interestingly, Ouchi and Matsushita (1992) reported that vertical exaggeration of the surfaces used in their analyses did not

significantly alter the fractal dimensions derived from those surfaces. Of course, the question of whether the dimension obtained using the divider method is theoretically meaningful must always be considered since its determination is a mechanical process which will always return a numerical value.

Clarke and Schweizer (1991) exaggerated the vertical component of some profiles and found that doing so changed little of their results. However, the largest exaggeration factor they used was only a ten times magnification—Brown (1987) used factors of up to 10^4, and only after the values had been exaggerated by a minimum of 10^2 were any significant changes noted. Furthermore, Clarke and Schweizer (1991) selected step sizes which spanned the crossover length, ensuring that the derived dimensions would always be close to one—the approximate value of their reported dimensions. Although they felt that their results served to cast doubts on Brown's work, the results actually lend some credence to the work. The dimensions reported from the ten times magnification were closer to the dimensions obtained from a variogram analysis of the surface than were the unmagnified dimensions.

Other problems with the divider method include deciding how many step increments are necessary to produce reliable results, and deciding the interval between divider widths. Since the fractal dimension is derived from the slope of the line obtained from the log–log plot of feature length vs. step size, theoretically a minimum of two points are all that are required. However, few people would consider the results of such an analysis reliable or representative, and it is usually suggested that between five and eight determinations be made (i.e., the step size be incremented five to eight times). Statistical reasoning suggests that evenly spaced values on the independent axis will improve the reliability of the estimates derived from least-squares regression. This requires that the divider widths double between each set of line determinations. However, these two concerns work against each other, since doubling the step size means that fewer length determinations can be made. This also means that small features (especially small islands) will tend to be excluded from the analysis, possibly producing an effect on the derived dimension similar to that reported above by Matsushita *et al.* (1991). This is one argument against using geometrically increasing step sizes.

The traditional divider method suffers from both practical and theoretical limitations. Because of this, the surficial divider method is a potentially more robust replacement for applications which utilize isolines derived from DEMs. This method is conceptually equivalent to measuring linear features from maps of various scales using a constant width divider. Starting with the isolines derived from the highest resolution data, their total length is determined. Then, the resolution of the data is reduced by selecting, for example, every second cell, and the isoline lengths are redetermined. The data subsampling process is repeated until the decimation is such that the nature of the original surface is

lost. The total length of the isolines at each determination is then used in the calculation of the fractal dimension. The "step size" for a given determination is obtained by dividing the total contour line length by the total number of coordinate pairs (minus one). The slope derived from the log–log plot of the total contour line length against the step size (average chord length) is then used to determine the fractal dimension—as in the traditional method (Table 2). This "surficial dividers" method was described in Paumgartner *et al.* (1981) as the variable magnification method.

After each resolution change, the possible number of data subsamples is increased by a factor of four. If all possible samples are used in the analysis, this should increase the method's reliability. This method does not suffer from the small islands problem, the single line behavior outlined by Matsushita *et al.* (1991), or from the crossover length problem. Klinkenberg and Goodchild (1992) found that this method produced dimensions similar to those produced by the variogram method when applied to the surface, while the more traditional divider methods (including several of the faster alternatives) produced much lower dimensions. When their data was reworked using the single contour line formula of Matsushita *et al.* (1991) the magnitude of the differences was reduced but not eliminated.

THE BOX COUNTING METHOD

The box counting method is widely used to determine the fractal dimension of many different phenomena (Table 1). Prior to its applications in fractal research, box counting was mainly used to quickly determine the area of irregular cartographic features (e.g., Gierhart, 1954; Maling, 1968). Since it can be applied with equal effectiveness to point sets, linear features, areas, and volumes, the box counting method is a widely used means of determining fractal dimensions. This method is also known as the grid or reticular cell counting method (Gagnepain and Roques-Carmes, 1986; Peitgen and Saupe, 1988), and has been shown to be equivalent to the Minkowski-Bouligond (or "sausage") dimension (Dubuc *et al.*, 1989b).

The basic implementation, using the determination of a linear feature as the example, is as follows:

1. Cover the feature with a single "box."
2. Divide the box into four quadrants, and count the number of cells occupied.
3. Divide each subsequent quadrant into four subquadrants, and continue doing so until the minimum box size is equal to the resolution of the data, keeping track of the number of quadrants or cells occupied at each step.

The fractal dimension is easily obtained from the log–log plot (Table 2).

The box counting method, when applied to linear features, is usually applied to cuts of a surface (e.g., contours) where the boxes overlay the cut lines. The method can be applied to profiles, however, as described below.

Divide the profile lengthwise into, say, four equal parts and count the number of intersections of a horizontal line at some specified vertical value (using only those values which occur at the section divides to determine if an intersection occurs). Then, increase the number of divisions and redetermine the number of intersections. Continue (geometrically) increasing the number of divisions and re-determining the number of intersections until the minimum resolution of the data is reached. Using a log–log plot of "box size" against the number of intersections, the fractal dimension is equal to the slope times minus one. Shelberg, Moellering, and Lam (1983) described a similar procedure for determining the fractal dimension of a surface, from which this procedure was abstracted.

Several problems have been identified with the standard implementation of the box counting method (Liebovitch and Toth, 1989). The method requires a significant amount of computer memory and computational time since a very large number of cells have to be stored. Because of this problem, Liebovitch and Toth (1989) introduced a "fast" algorithm which, using a statistically-based sampling approach, does not require a complete enumeration of every cell at the higher resolutions. Their method is best applied only to datasets of low fractal dimensions, however, Sarraille (1991) has produced an implementation of their method.

Box counting also requires a large number of data points in order to produce correct dimension—Dubuc *et al.* (1989a) reported instabilities in the method when the number of data points used was small. They also found that the method was sensitive to the level of quantization of the data. This sensitivity may apply to other methods, and should be investigated.

The question of defining the minimum and maximum box size was addressed by Liebovitch and Toth (1989). The first two box counts (i.e., when one and four boxes are used) should not be used in the slope determination, nor should the box counts which occur when the cell size approaches the resolution of the data (i.e., when each data point falls in a single box). Dubuc *et al.* (1989a) identified problems with the stability of the slope when the larger box sizes were included in the analysis. As with every other method which determines the slope in log–log space, the cell sizes should change as a function of a power of two so that they will be evenly spaced in the log space. Dubuc *et al.* (1989a) found that doing so too rapidly covered the data, resulting in too few points in the log–log plot. Therefore, they did not use dyadic boxes.

Box counting can also suffer from a "remainder" problem. If the boxes cannot cover the data evenly (e.g., if the raw data consists of a 217 by 217 array), then some cells will be missed if the box sizes increase geometrically.

Because of this potential problem, some authors suggest first mapping the raw data onto a square unit. However, if the raw data was collected on a non-square frame (e.g., a 149 by 217 array), then that option will not be available. Very little has been written about this problem in the box counting literature.

Lovejoy, Schertzer, and Tsonis (1987) present a detailed discussion of the box counting method and note that the boxes need not be square. By using rectangular boxes, for example, non-isotropic, multidimensional fractals can be studied. However, discussion of such extensions to the basic method are outside of the scope of this paper, however.

SPECTRAL METHODS

Using spectral methods to obtain the fractal dimensions of features is another widely used method (Table 1). Although a rigorous method, it is computationally difficult and computer intensive (Bartlett, 1991). This method requires much more data preprocessing than any of the other methods, all of which work with the data "as is." Spectral methods require the raw data to be detrended and tapered, and not doing so properly can greatly alter the results (Fox and Hayes, 1985; Carr and Benzer, 1991). They should only be applied to self-affine curves (i.e., profiles) since the method will always return a fractal dimension of one for self-similar curves (Peitgen and Saupe, 1988). Descriptions of the steps required to perform a spectral analysis for fractal purposes can be found in Huang and Turcotte (1989, 1992), Pentland (1984), Peitgen and Saupe (1988), and Turcotte (1992).

An early application of the power spectrum method to the study of surface topography was reported in Sayles and Thomas (1978). Berry and Hanny (1978) subsequently placed those results into a fractal framework. It was not until Berry and Lewis (1980) that a formal link between the fractal dimension and the power spectrum was provided, however. Finally, Mandelbrot *et al.* (1984) cleared up some of the practical issues related to the method which had arisen in the literature.

Power and Tullis (1991) mention several problems with applications of spectral methods. One problem is that the typical frequency progression used in FFT algorithms is arithmetic. This results in the higher frequencies being over-represented—relative to the lower frequencies—in the log-log plots used to determine the fractal dimension (Table 2). Other methods are easily adjusted to produce equally-spaced data points on the independent axis, which results in more stable least-squares parameters. A more significant problem is that several authors have made incorrect assumptions about the appropriate slope of the power spectrum plot, perhaps because there are a variety of ways of expressing the power or amplitude spectra (Power and Tullis, 1991; also see Carr and Benzer, 1991). Dubuc *et al.* (1989a) reported lower precisions from power

spectrum methods when compared to their other results. They found that the points in the log–log plot rarely formed a straight line, increasing the instability of the least-squares derived values.

The computational complexity of spectral methods is well illustrated by the series of papers by Huang and Turcotte (1989, 1990) and Goff (1990). In their first paper, Huang and Turcotte (1989) reported that the fractal dimensions of one-dimensional profiles were significantly lower (0.5) than the fractal dimensions of the surfaces (minus 1) as determined using two-dimensional FFT. Subsequently, an error in their formulation was noted (Goff, 1990). Huang and Turcotte (1990) re-examined their original work and reported that the dimensions (plus one) produced by the one-dimensional FFT were within statistical uncertainty of the two-dimensional FFT values. Thus, it appears that this method produces consistent fractal dimensions when applied to surfaces with differing topology dimensions. Nonetheless, the inherent complexity of spectral methods requires that they be carefully implemented—Power and Tullis (1991) found several examples where subsequent re-interpretation of previously published spectral analyses have dramatically altered the conclusions.

THE VARIOGRAM METHOD

The variogram method is widely used in the determination of the fractal dimension of surfaces (Burrough, 1981, 1984) and appears to have properties—in particular its ease of use—which make it a preferred method over spectral analysis (Carr and Benzer, 1991; Klinkenberg and Goodchild, 1992). Although it has been much less commonly applied to strictly linear phenomena, the method is very easy to implement when analyzing self-affine profiles. By sampling a large number of pairs of points (of differing spacings) along the profile and computing the differences in their vertical values (e.g., z values) the fractal dimension is easily derived from the log–log plot of (expected differences in $z)^2$ vs. distance between the point pairs (Table 2).

The choice of the maximum point-pair distances used in the analyses requires some thought. The maximum distance between point pairs is usually taken to be one-half of the absolute maximum distance between points, although some authors have suggested considering much shorter maximums, such as one quarter the maximum distance (Roy, Gravel, and Gauthier, 1987). However, this rule may be more restrictive than necessary. Very linear relations extending out to absolute maximum distances have been noted (Klinkenberg and Goodchild, 1992). More often, however, the dimensions associated with distance pairs that are greater than the suggested cutoff are larger than those associated with the distance pairs that are less than the suggested cutoff (Mark and Aronson, 1984; Klinkenberg and Goodchild, 1992). The shortest point-pair distance used in the analyses will depend on the resolution of the data.

For statistical purposes, it is important that the sample of point pairs used in the analyses uniformly span the distance range. In order to obtain statistically valid averages for the z differences, the point-pair distances are usually placed in a number of defined classes. This means that, as with the other methods which use linear regression in the determination of the slope, the distance classes should be selected so that they are evenly spaced in the log space used in the least-squares regression. Scatter of the data points in the log–log plot can cause some instabilities in the least-squares derived parameters.

This method does appear to produce consistent fractal dimensions when applied to features with differing topological dimensions. Klinkenberg and Goodchild (1992) found that the variogram-derived fractal dimensions of profiles were equal to the fractal dimension of the surfaces. Lovejoy and Schertzer (1987) criticized the variogram method, stating that the method only explores the scaling nature of the vertical fluctuations, and does not incorporate the scaling of horizontal structures. Although the variogram is a function of differences in, usually, elevation, and horizontal distance, one could imagine a "flipped" variogram, wherein the vertical differences become the independent variable and the average (horizontal differences)^2 associated with each vertical difference the dependent variable. From this perspective their criticism does warrant some consideration.

AREA-BASED METHODS

Two area-based methods can be used to determine the fractal dimension of linear features if those features form closed loops (Mandelbrot, 1975). The area-perimeter relation and Korcak's empirical relation for islands both have been used to determine the fractal dimension of lakes, contour loops, and islands (Table 1). If the data is appropriate, these are relatively simple methods to use and their implementation is fairly simple, requiring few of the decisions most of the other methods demand (Table 2). When using the area-perimeter relation, however, one must not confound the relation by using perimeters derived from sources of different scales (i.e., the area of a given feature will not change much when measured from sources of different scales, but the perimeter measurements can change significantly).

The requirement that the features form closed loops restricts the usage of area-based methods. For example, although all contours must eventually loop back upon themselves, they may not do so within the particular study area. Thus, few objects may be available for analysis. However, these methods should be fairly robust, since they will not suffer from the problems identified above with respect to the whole vs. parts dimension, or from the small islands problem. Sakellariou *et al.* (1991) found the area-perimeter method to be the most stable and least ambiguous method of the four they considered (which included var-

iogram and spectral methods). Mandelbrot *et al.* (1984) found the dimension obtained from an area-perimeter analysis to agree with the dimension obtained from a spectra analysis.

THE LINE SCALING METHOD

One essential aspect of fractal features is their scaling nature. That means that the moment statistics of a fractal feature are (theoretically) unstable (Mandelbrot, 1982). For example, the longer the topographic profile, the greater the observed variability in elevations (Ahnert, 1984). Building on that concept, Matsushita and Ouchi (1989) developed a method for the determination of the fractal dimension of a linear feature which used the observed relationship between sample sizes and sample variances (Table 2).

The method is conceptually very simple to implement and works on both self-similar and self-affine curves. Samples of various lengths (say N) are taken from the curve and the standard deviations of the coordinates along each axis are determined (say X and Y). From the two log–log plots of standard deviation (X and Y) vs. sample size (N) the respective slopes are determined (say V_X and V_Y). If V_X and V_Y are equal or nearly so, that is a good indication that the curve is self-similar. For these curves the fractal dimension is computed as $D = 1/V_X = 1/V_Y$. If V_X and V_Y are unequal—a good indication that the curve is self-affine since the coordinates scale differently—the two variances are related to each other via the Hurst scaling parameter (H) as $H = V_Y/V_X$. Matsushita and Ouchi (1989) tested their method using linear features of known fractal dimensions (both self-similar and self-affine) and found the results to be very reasonable.

This method has yet to be tested extensively, but it would appear to have characteristics which could make it a useful tool in fractal analysis given that it works with both self-similar and self-affine features. Furthermore, Ouchi and Matsushita (1992) have extended this method to handle surfaces and reported good results.

MISCELLANEOUS METHODS

There are several additional methods which require some discussion. The intersegment angles method introduced by Eastman (1985) is based on relating local measures of a curve's sinuosity to its fractal dimension. Eastman analyzed several well known natural curves (e.g., the west coast of Britain) and obtained results that matched those reported in the literature. The method was conceived by observing the relation between the characteristic generators of systematic fractal curves, their fractal dimension, and their sinuosity. Extending the idea

of a generator to random fractals, Eastman (1985) developed the formula below. It directly related the sinuosity of a natural curve to its fractal dimension.

$$D = \frac{\log (2)}{\log \left(2 + \log \left[\left\langle \left(\frac{c}{a + b} \right)^2 \right\rangle \right]^{0.5} \right)}$$

a, b = segment lengths about point j (i.e., a and b will span lengths from pt (j) to pt $(j \pm k)$, where k = 1 to 5), and c = distance from pt $(j - k)$ to pt $(j + k)$.

The variation method, recently developed by Dubuc *et al.* (1989a), is a variation of the Minkowski method. Instead of using fixed disks (or boxes) to cover the feature, they use an adaptive covering procedure—boxes with dimensions that are a function of the curve within a local neighborhood. The optimal size of the local neighborhood is chosen by observing the relation between the data point scatter and the neighborhood size. Details on how to implement the variation method are presented in Dubuc *et al.* (1989a). The method can also be applied to surfaces (Dubuc *et al.*, 1987, 1989b).

Tests of the method—using mathematically defined random fractal curves—produced dimensions almost identical to the curves' theoretical dimensions. However, other methods shown to produce accurate dimensions when working with well-behaved fractal curves have not always behaved as well when applied to "real world" fractals. It remains to be seen how the variation method works when applied to natural datasets. Another variant of the box counting method was presented in Rigaut (1991). It too requires further testing.

In the study of diffusion-limited aggregation (DLA), the mass fractal dimension is commonly determined. A comprehensive review of DLA and determination of the mass dimension was presented by Meakin (1991). The methodology is easily adapted to linear phenomenon: plot the (log of the) accumulated length $L(R)$ of the feature as a function of the (log of the) distance (R) from a central point (D = the slope). It has been little used in areas other than DLA, but the method will probably play a much larger role in the future as fractal analyses in the geosciences become more sophisticated (e.g., Bartoli *et al.*, 1991). Batty and Longley (1987) and Benquiqui and Daoud (1991) provide atypical examples of the application of this method.

DISCUSSION AND CONCLUSIONS

Some general observations can be made about the methods described above. While every method appears to have its special concerns—aspects of their particular application which must be carefully considered before results are reliably produced—many share common concerns. Using least-squares analysis to de-

termine the slope of the log–log curve means that the selection (i.e., the range and spacing) of the independent variable values can greatly influence the results. The autocorrelated nature of the data should also be considered, but rarely is (Reeve, 1992). Another aspect which hasn't been mentioned in the fractal literature is the lack of correspondence between parameters obtained from nonlinear regression and those obtained from linearized log–log regression. That the values produced by the two forms of regression can be very different has long been known (e.g., Zar, 1968). In island biogeography, another field which similarly uses log–log regression to determine meaningful parameters, it has been shown that differences in the interpretation of supposedly meaningful parameters can be traced back to the linearization of the underlying power relation (Klinkenberg, 1988). The effect of this on the interpretations of fractal dimensions requires exploration since the effects of linearization may not be constant from method to method.

For most methods the choice of the smallest (and largest) sample elements requires careful consideration. Reducing the band width within which the particular method is employed will necessarily reduce the likelihood that fractal elements be identified, if present, or that multifractal behavior be identified. Furthermore, imposing a dyadic sampling strategy for regression purposes can result in significant reductions of the sample size.

There has been a range of contrasting opinions regarding the choice of one method over another. Huang and Turcotte (1989) prefer spectral methods over methods such as the variogram method because the intercept serves as a quantitative measure of texture. However, this is not really a reason for using the method because the variogram intercept can be used in a similar fashion (Klinkenberg and Goodchild, 1992). Several authors have commented that they feel that the variogram method is more robust/accurate than the power spectrum method, and is better able to detect the presence of fractal elements (e.g., Malinverno, 1989; Klinkenberg and Clarke, 1992). However, variogram methods, since they are based on second-order statistics, may not always be able to detect breakdowns in the scale invariance of phenomenon—spectral methods may be better able to detect such breakdowns (Jones, Thomas, and Earwicker, 1989). With appropriate data, Sakellariou *et al.* (1991) found the area-perimeter method the most preferable.

Brown (1987) found that the spectral and divider methods yielded similar dimensions for self-affine profiles provided that the divider method was used properly (i.e., the crossover length was respected). On the other hand, Carr and Benzer (1991) feel that there is no prerequisite that the two methods yield similar fractal dimensions since the divider method is based on geometric relations, while the spectral method (as with most other methods) is based on stochastic relations. Furthermore, since spectral methods are applied to profiles, while dividers are most often applied to horizontal cuts, there are many reasons why

the two methods produce different dimensions. Much of the controversy in the fractal literature derives from the observation that different methods often yield different and sometimes ambiguous dimensions. It should be apparent that, in some cases, the differences may be the result of inappropriately applied methods. On the other hand, some differences should be expected, since the methods are not all measuring the same fractal quantity.

No one method appears to be "the" method with which to determine the fractal dimension of linear features. Some of the methods have only recently been introduced (e.g., the line scaling and the variation methods) and they need to be applied to a variety of natural features before any conclusion is reached about their overall reliability and consistency. Some methods can be applied to both self-similar and self-affine curves, while others should only be applied to one type or the other. The particular choice of a method should depend on a number of factors, some of which are outlined below.

1. Whether the feature is self-similar or self-affine.
2. The format in which the data will be provided, which can make certain methods much easier to implement than others.
3. Whether features from a range of topological dimensions will be investigated, in which case one of the extensible methods—such as box counting, variogram, or spectral methods—may be the preferred choice.
4. Which method fits most comfortably within the current methodological framework of the researcher.
5. Whether multidimensionality is known to exist, in which case the box counting method as extended by Lovejoy *et al.* (1987) should be employed.

The main conclusion to be reached is that while there are many ways of determining the fractal dimension of linear features, the application of every method requires careful consideration of a range of methodological concerns. Without adequate consideration of the potential problems the results from any analysis will not truly reflect the fractal nature of the feature.

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