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Fractal Structure and Statistics of Computer-simulated and Real Landforms

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A vertical profile of landform looks similar to the one-dimensional Brownian motion trace. It is regarded as a self-affine curve characterized by the Hurst scaling exponent, H ($0 < H < 1$). The fractal dimension D_e for the pattern of entire contour lines including all islands, and D_c for a single contour line can be both calculated from H as $D_e = 2 - H$ and $D_c = 2/(1 + H)$. The size distribution of islands follows the power-law (the Korcak's law) with the exponent, ζ, characterized by D_e as $\zeta = D_e/2$. We confirmed these scaling laws both on computer-simulated landforms and real coastlines, although the former is dependent on the system size and the latter includes disturbing effects of coastal processes. The value of H, which expresses a relief characteristic of the three dimensional self-affine surface such as landform, can be calculated from a horizontal section by these three scaling laws.

KEYWORDS: fractal, self-affinity, Hurst exponent, landform, Korcak's law
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1. Introduction

A vertical profile of landform looks very similar to the one-dimensional Brownian motion trace. That is to say, both are self-affine curves scaled anisotropically and characterized by Hurst exponent, H.[1,2] In general, a self-affine shape is said to "have a persistence" for $H > 0.5$, or "have an anti-persistence" for $H < 0.5$.[1] Many phenomena in nature form self-affine shapes, such as landform,[3] fractured surface of rocks, minerals and metals, growing interfaces of paper wetting[4] and bacterial colonies.[5] An analysis on the self-affine shapes, therefore, may provide a useful tool for the numerical treatment of complex natural phenomena.

Matsushita *et al.* revealed that the fractal dimension D_e for the whole contour lines of a certain height including separated islands and the fractal dimension D_c for a single continuous contour line (coastline) depend on H as $D_e = 2 - H$ and $D_c = 2/(1 + H)$, respectively.[6,7] They also revealed that the power-law exponent, ζ, which characterizes the size-distribution of islands, is expressed as $\zeta = D_e/2$.[8] The complex natural structure of landform seems able to be expressed by these simple scaling laws. In this study, we examine these relations both on the computer-simulated landform and on the real landform.

2. Computer-Simulated Landform

Landforms specified by the value of H are generated with using the random addition midpoint displacement method.[9] Several different landforms are generated for each value of H, and 20 sections of contour lines are taken for the analysis. This process is repeated with different grid numbers (system sizes), 256×256, 512×512, and 1024×1024. The fractal dimension of each contour line is estimated by the box-counting method, excluding peripheral 10% to avoid the boundary effects. The longest continuous contour line (coastline) is picked up to estimate D_c. Figure 1 shows some examples of generated contour lines for three different system sizes at $H = 0.5$. Contour lines of larger system size obviously reveal finer details. The results of analysis in the case of system size 1024×1024 are shown in Fig. 2. The values of H estimated from those contour lines are close to the specified value (0.5) of H.

2.1 *Entire contour lines of a certain height including islands*

Figure 3(a) is a graph showing the relationship between the specified H and the estimated D_e for the whole contour lines of a certain height including separated islands. The line showing the theoretical relationship between H and D_e fits the plot very well. Especially, in the region where H has anti-persistence, the fit becomes better as the system size increases. This means that the scaling law $D_e = 2 - H$ evidently holds.

2.2 *A single continuous contour line*

Obtained values of D_c are shown in Fig. 3(b) as a relation with the specified H. In the region where H has anti-persistence ($H < 0.5$), the D_c values become closer to the theoretical values calculated from the scaling law $D_c = 2/(1 + H)$ and specified H as the system size increases. On the other hand, in the region where H has persistence ($H > 0.5$), the value of D_c seems to be independent of the system size. This discrepancy may probably be derived from the inadequacy of computer simulation algorithm or the box-counting method employed in this study. However, the results show a good fit in the area nearby $H = 0.5$, in which the H values of real landscapes are commonly found.

2.3 *Cumulative number of islands*

Figure 4 shows the relationship between the H_c values calculated from ζ and the specified H. Figures 4(a), 4(b), and 4(c) correspond to three system sizes, 256×256, 512×512, and 1024×1024, respectively. The values of H_c on vertical axis are calculated as $H_c = 2 - 2\zeta$, where $D_e = 2 - H$ and $\zeta = D_e/2$. The values of H_r ($H_r = 2 - D_e$) are also plotted as a reference.

The calculated values of H_c and H_r generally well re-

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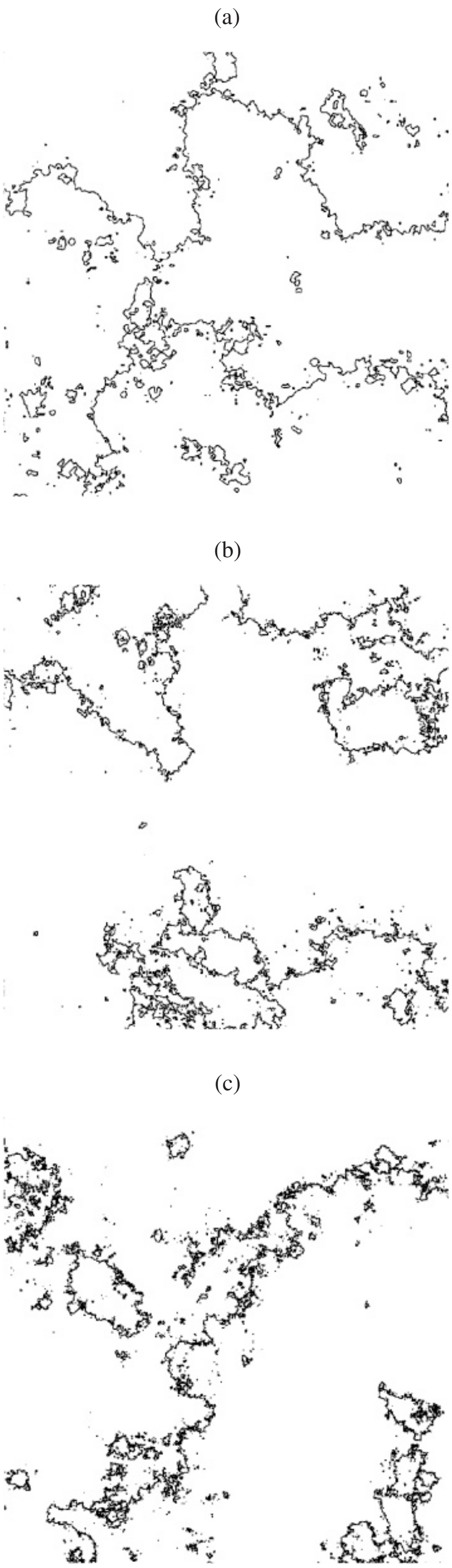


Fig. 1. Figures showing the entire pattern of contour lines of a certain height generated by the random addition midpoint displacement method. The specified value of H is 0.5. The system sizes of (a), (b), and (c) are 256×256, 512×512, and 1024×1024, respectively.

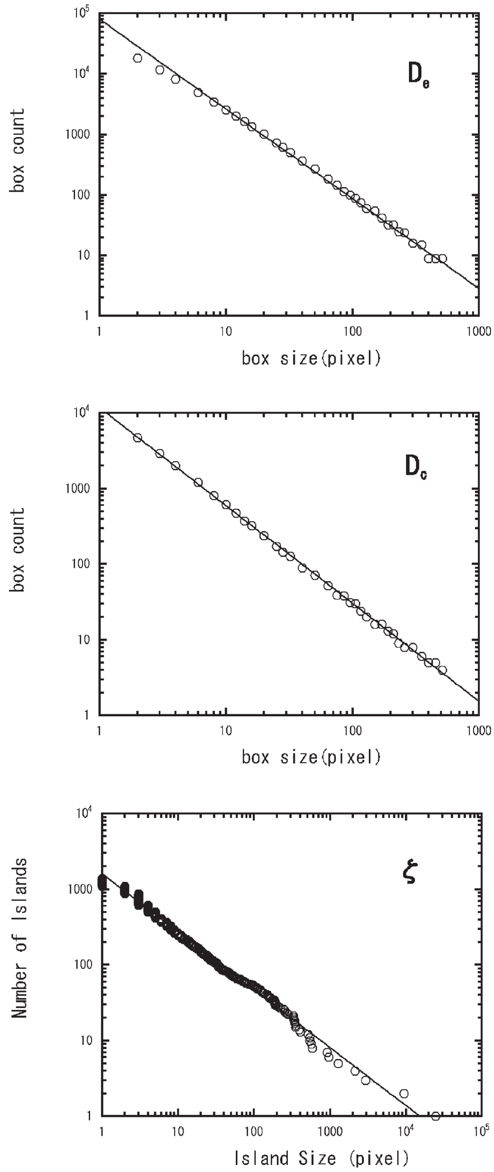
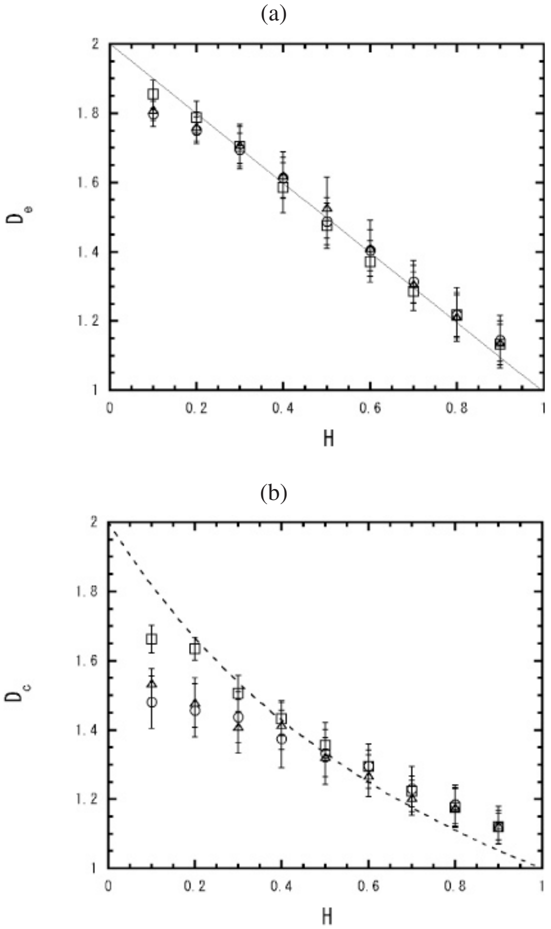


Fig. 2. The results of fractal analysis on Fig. 1(c) by the box-counting method. The base plots for the calculation of D_e, D_c, and ζ are shown, respectively. The calculated values are $D_e = 1.483$, $D_c = 1.294$, and $\zeta = 0.766$, with $H = 0.517$, 0.546, and 0.468, respectively.

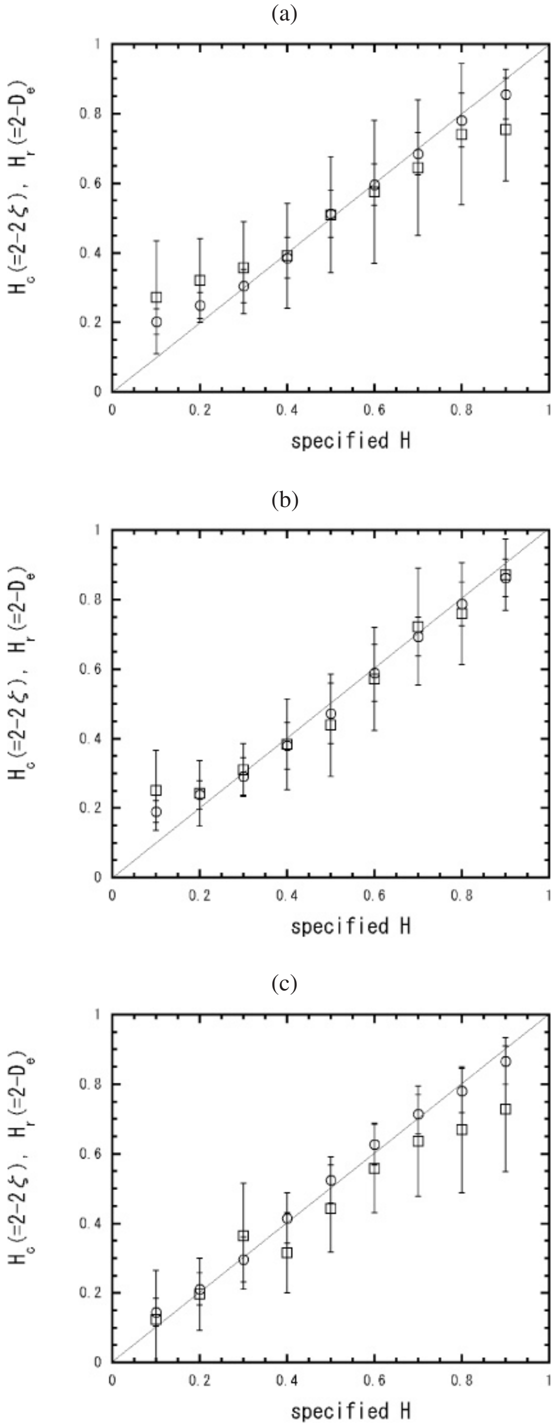
produce the values of specified H. This implies not only that island's size distribution is explained by a power-law known empirically as the Korcak's law, but also that the exponent is expressed by the scaling law ($\zeta = D_e/2$). In Fig. 4(a) with the system size of 256×256, the error bars for the dispersion of H_c are longer than in the cases of larger system sizes. And in the areas of small and large specified H, namely $H = 0.9$, 0.3, 0.2, and 0.1, the H_c values show significant displacement from the values of specified H. On the other hand, with the system size of 512×512 [Fig. 4(b)], the error bars are shorter and the values of H_c are nearly equal to the specified values except at $H = 0.1$.

In the case of 1024×1024 system size [Fig. 4(c)], however, the mean values of H_c do not well reproduce the specified values except at $H = 0.1$ and 0.2. This discrepancy is probably derived from the resolution of the image, or the number of pixels, on which box-counting is performed. In this study, the image resolution for the analysis is 1104×1104 pixels. Expressing the 1024×1024 grids by the resolution of 1104×1104 pixels may inevitably cause ambiguity in the representation of details. The computer program developed in this study to measure the size distribution of islands counts islands sharing only one pixel



○ : 256 × 256, △ : 512 × 512, □ : 1024 × 1024

Fig. 3. Graphs showing the relationship of D_e (a) and D_c (b) to specified H at each system size. The solid line in (a) and the broken line in (b) are lines of $D_e = 2 - H$ and $D_c = 2/(1 + H)$, respectively.



○ : H_r, □ : H_c

Fig. 4. Graphs showing the values of H calculated from the obtained values of D_e and ζ as $H_r = 2 - D_e$ and $H_c = 2 - 2\zeta$ in relation to the specified values of H for three system sizes. The system sizes of (a), (b), and (c) are 256 × 256, 512 × 512, and 1024 × 1024, respectively.

as one island. Fine resolution, therefore, is important for this analysis. The system size of 256 × 256 seems too small, and the resolution (1104 × 1104 pixel) seems too coarse for the system size 1024 × 1024. On the other hand, the excellent fit of H_c to specified H with the system size 512 × 512 (the resolution is approximately 900 × 900 pixels) indicates the good combination of system size and resolution. Similarly, Robert and Roy pointed out that the fractal dimension of a stream basin is affected by map-scales through the process of cartographic abstraction.[10)]

3. Coastline of the Kujukushima Area, Nagasaki, Japan

We analyzed the coastline of the Kujukushima area, Nagasaki, western Japan. This area has complex saw-toothed coastlines and many small islands of various sizes. The coastline is picked up from 1 : 70,000 digital elevation maps "ProAtlasW3" published by Alps, Inc., Nagoya. We divided the area into north- and south-Kujukushima and analyzed them separately in order to see the possibility of local variation.

Figure 5(a) shows the entire coastline of the Kujukushima area and (b) the longest single coastline of the area. We measured fractal dimensions of the entire coastlines (a) and the longest single coastline (b) by the box-counting method, and estimated Hurst exponents from the fractal dimensions. The results are compiled in Table I and the examples of box-counting are shown in Fig. 6.

Fractal dimensions D_e and D_c obtained for three areas (entire area, north and south areas) are almost identical. This clearly expresses the fractal characteristic of landform in the area. The values of D_c obtained here are close to the value of fractal dimension generally considered to represent a saw-toothed coastline (approximately 1.3). In addition,

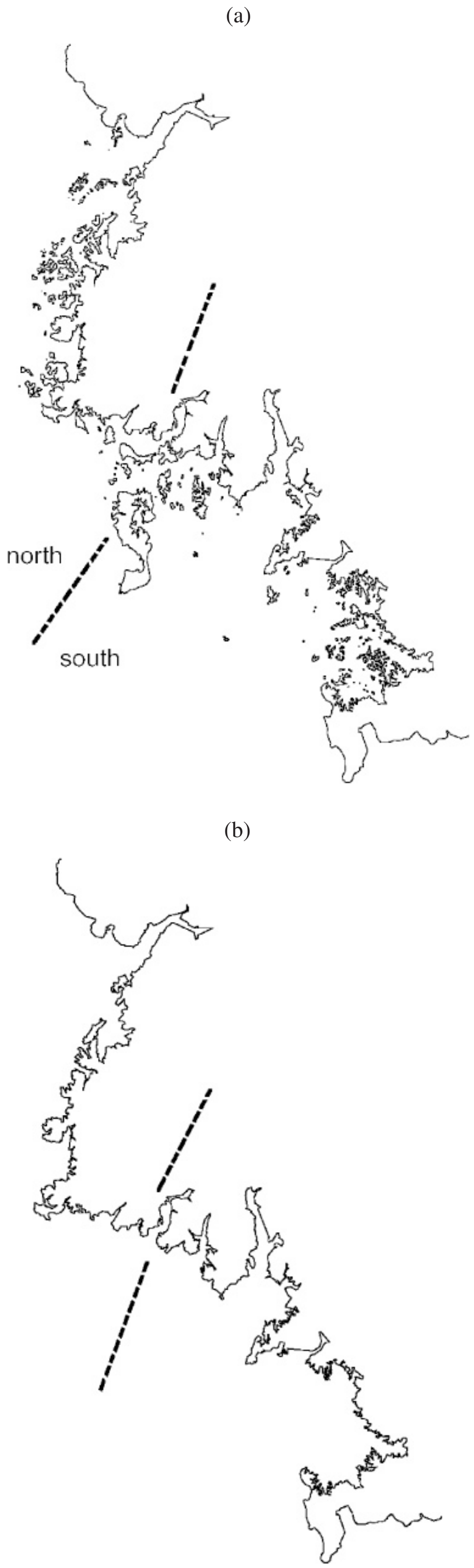


Fig. 5. Patterns of coastlines including all islands (a) and the longest continuous coastline (b) in the Kujukushima area, Nagasaki, Japan.

Table I. Results of fractal analysis on the coastlines in the Kujukushima area.

Area	North-Kujukushima	South-Kujukushima	Entire Kujukushima
D_e	1.427	1.436	1.440
D_c	1.257	1.295	1.282
H from D_e	0.573	0.564	0.560
H from D_c	0.591	0.544	0.560
ζ	0.714	0.718	0.720

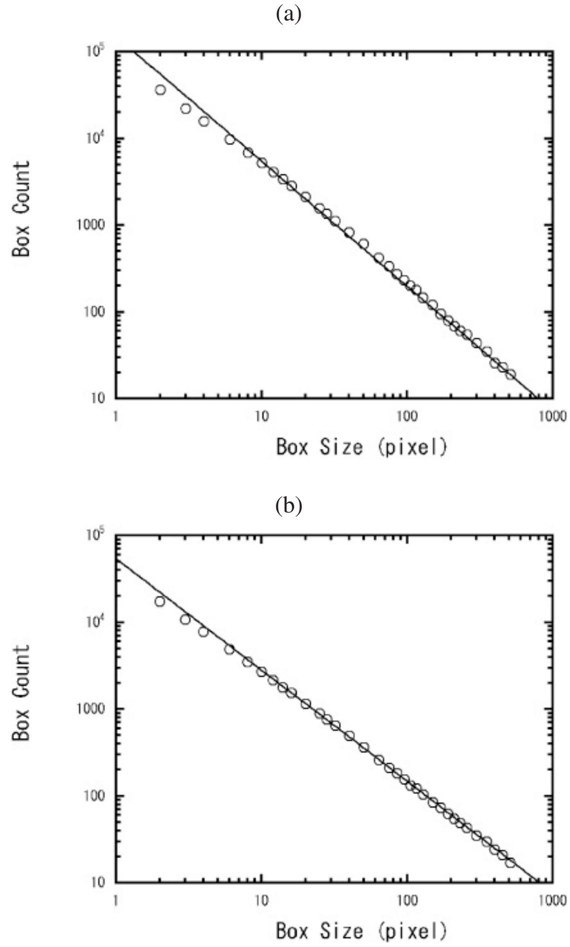


Fig. 6. Results of box-counting on Figs. 5(a) and 5(b) in the Kujukushima area. Obtained values are $D_e = 1.440$, $H = 0.560$, and $D_c = 1.282$, $H = 0.560$.

calculated values of the Hurst exponent are all similar to each other. These consequences indicate that the scaling laws, $D_e = 2 - H$, $D_c = 2/(1 + H)$ and $\zeta = D_e/2$, evidently hold.

We also analyzed the distribution of cumulative number of islands classified by size in this area (Fig. 7). The cumulative number plotting is examined with the values of ζ calculated from D_e (Table I). The broken line in Fig. 7 is the line indicating the dominant trend of distribution. Although the slope of the line (~ -0.60), the absolute value of which indicates the value of ζ, is somewhat different from the value of ζ calculated from D_e of the coastline (0.72; solid line), the graph confirms that the island-size distribution follows the power-law (Korcak's law). The difference in the values of ζ may come from the disturbing effects of artificial deformation and coastal processes on the coastline. The difference may be in the range of error because the error bars for the cumulative number plotting are relatively long even on the computer generated landforms (Fig. 4).

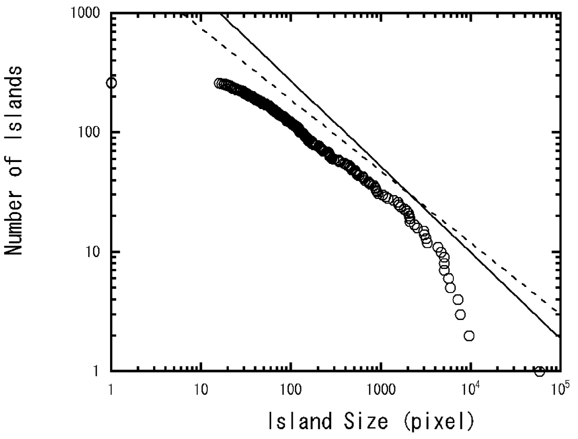


Fig. 7. The cumulative number of islands for their sizes in the Kujukushima area. The broken line is the line indicating the dominant trend of distribution, and the slope is ~ -0.60. The solid line is the same kind of line for the size distribution of islands calculated from D_e value of contour lines and the slope is ~ -0.72.

4. Discussion and Conclusion

Our analysis revealed that the scaling laws about D_e and ζ well hold for the computer-generated coastline and island-size distribution, although the value of ζ is affected by image resolution and system size. The measured fractal dimension, D_c, becomes closer to the specified value as the system size increases in the region of anti-persistence, whereas in the region of persistence the measured values of D_c shift characteristically away from the theoretical values independent of system size. In other words, the scaling laws well hold on larger system size in the region of anti-persistence, whereas they are independent of system size in the region of persistence. In the region of anti-persistence where fine patterns are dominant, the characteristic of the finer pattern emerges as the system size increases, and the measured values of D_e, D_c, and ζ become close to the theoretical values. In the region of persistence where coarse patterns are dominant, on the other hand, the increase in system size does not seem to affect measured values. The picture looks almost unchanged as the system size increases in the region of persistence, while it looks different in the region of anti-persistence.

The values of H obtained for the Kujukushima area revealed the fractal nature of real coastlines. The values of H calculated from D_e and D_c are nearly identical; therefore, we conclude that the 2-dimensional pattern of contour lines is characterized by the Hurst exponent (H) which expresses the relief texture of 3-dimensional landform. On the other hand, the power-law exponent of the size distribution of islands, which is supposed to be ζ, differes from the value of $D_e/2$ obtained for the coastline, although the size distribution satisfies the Korcak's law. The real island size and coastlines may reflect more complex processes of landform formation than the computer simulated landform.

Contour lines of erosion landform have self-affine nature because landform formed by fluvial erosion is considered to be self-affine, while coastlines and island size may have effects of coastal processes and artificial deformation in addition to the fluvial erosion. The value of H calculated from a coastline, therefore, may differ from the value of H calculated from a landscape profile in the same area. This difference in H values, if any, can provide some information on coastal processes.

We confirmed that the three scaling laws

$$D_e = 2 - H, \quad D_c = \frac{2}{1+H}, \quad \zeta = \frac{D_e}{2}$$

mostly hold both on horizontal sections (contour lines) of computer-simulated landforms and real landforms, while in the case of computer simulations their validity is dependent on the system size, and real coastlines inevitably include disturbing effects of coastal processes and artificial deformation. The value of H calculated from these scaling laws on horizontal sections represents the H value of the three-dimensional surface. This method would make it easier to analyze the characteristics of three-dimensional relief.

In §2.2, we could not give a precise explanation for the discrepancy between our results and the theoretical result. It is our future task that we analyze a landform generated by other algorithms and calculate the fractal dimension by other methods.

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