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A diffusion-limited aggregation model for the evolution of drainage networks

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ABSTRACT

We propose a modified diffusion-limited aggregation (DLA) model for the evolution of fluvial drainage networks. Random walkers are introduced randomly on a grid, and each two-dimensional random walk proceeds until the walker finds a drainage network on which to accrete. This model for headward growth of drainage networks generates drainage patterns remarkably similar to actual drainages. The model also predicts statistical features which agree with actual networks, including the frequency–order (bifurcation) ratio ($R_b = 3.98$) and the stream length–order ($R_r = 2.09$). Using the definition of network fractal dimension $D = \log R_b/\log R_r$, we find that our DLA model gives $D = 1.87$, near the observed range of $D \approx 1.80$–1.85.

1. Introduction

At short and intermediate length scales, the topography of the earth is dominated by erosional features. While crustal and lithospheric processes generate large-scale mountain belts, erosional dissection generates relief over smaller length scales. One of the remarkable features of this process is the self-organization of topography into a scale-invariant (fractal) form, such that the aspect ratio of topography is roughly constant regardless of amplitude [1]. The fact that topography worldwide exhibits the same fractal dimension regardless of climate, tectonic setting or bedrock lithology suggests that this self-organization reflects the most fundamental mechanics of erosion.

The evolution of drainage networks is intimately tied to the erosion of landforms. While hillslopes and interfluves (with length scales of less than 1–10 km) represent sites of relatively slow diffusional mass transfer (i.e., mass wasting), channels represent sites where the water flux becomes sufficient to promote a more efficient, advective mass transfer [2]. It follows that the generation of erosional relief is dominated by mass transfer in channels. Therefore, to understand the self-organization of topography, one must understand the spatial and temporal evolution of drainage networks. The statistics of these networks satisfy a variety of power laws [3,4], including frequency–length statistics and drainage area–length statistics. Thus, it is likely that the fundamental processes responsible for generating fractal stream networks are also responsible for generating fractal landforms.

In this paper we first discuss fractal networks in general, and drainage networks in particular. We then present a modification of the diffusion-limited aggregation (DLA) approach to drainage network generation. This results in networks that are qualitatively and quantitatively similar to actual networks.

2. Fractal trees and river networks

Drainage networks are now recognized as classic examples of fractal trees [5]. Three examples of fractal trees are given in Fig. 1. In order to specify the geometries, three quantities must be given: the branching ratio N_n/N_{n+1} where N_n is the number of branches of order n, the length ratio r_{n+1}/r_n where r_n is the length of the branch of order n, and the angle of divergence θ. For the

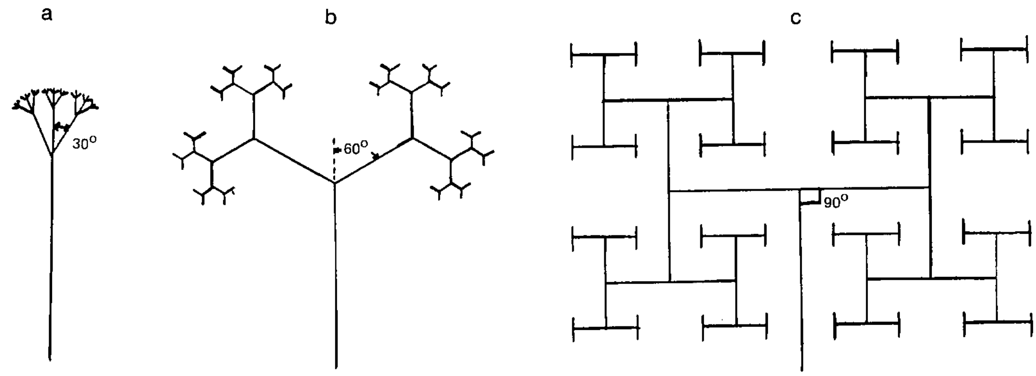

Fig. 1. Three examples of fractal trees. (a) $N_n/N_{n+1}=3$, $r_{n+1}/r_n=3$, $\theta=30°$, $D=1$. (b) $N_n/N_{n+1}=2$, $r_{n+1}/r_n=2$, $\theta=60°$, $D=1$. (c) $N_n/N_{n+1}=2$, $r_{n+1}/r_n=\sqrt{2}$, $\theta=90°$, $D=2$.

example given in Fig. 1a, $N_n/N_{n+1}=3$, $r_{n+1}/r_n=3$, and $\theta=30°$. Taking the definition of the fractal dimension D to be

$$D=\frac{\ln(N_n/N_{n+1})}{\ln(r_{n+1}/r_n)} \tag{1}$$

we find $D=1$ for this network. For the example given in Fig. 1b $N_n/N_{n+1}=2$, $r_{n+1}/r_n=2$ and $\theta=60°$, and again $D=1$. And for the example in Fig. 1c, $N_n/N_{n+1}=2$, $r_{n+1}/r_n=\sqrt{2}$ and $\theta=90°$, and D = 2. In all cases the constructions can be extended to infinite order without overlap. If the construction in Fig. 1c is extended to infinite order the plane is entirely covered by the construction, but with no overlap. Thus, this construction is an example of a self-similar (identical at all scales), deterministic network that can drain every point on a surface at as small a scale as is specified. This is the implication of $D=2$, the dimension of a plane.

It is standard practice to use the Strahler [6] stream-ordering system outlined in Fig. 2 to order actual drainage networks. When two like-order stream segments combine, they form a downstream segment one order higher than the original. Thus, two first-order streams combine to form a second-order stream, two second-order streams combine to form a third-order stream, and so forth. The bifurcation ratio R_b was defined in [7] as:

$$R_b=\frac{N_n}{N_{n+1}} \tag{2}$$

where N_n represents the the number of streams of order n. The length–order ratio R_r is defined by:

$$R_r=\frac{r_{n+1}}{r_n} \tag{3}$$

where r_n represents the mean length of stream of order n. Empirically, both R_b and R_r are found to be nearly constant for a range of stream orders in any given drainage basin. These are known as Horton's laws. Combining (1), (2) and (3) gives:

$$D=\frac{\ln R_b}{\ln R_r} \tag{4}$$

Standard stream-ordering parameters are directly related to the fractal dimension of the network [8].

An actual example of a drainage network is given in Fig. 3, from the Volfe and Bell Canyons in the San Gabriel Mountains near Glendora, California [9]. The smallest streams appearing on this map are one order lower than the smallest streams on the published topographic map [9]; we refer to these as 0 order streams. The number–length statistics for this network are given in Fig.

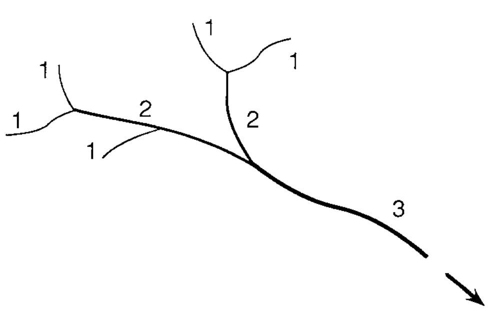

Fig. 2. Illustration of the Strahler [5] stream ordering system.

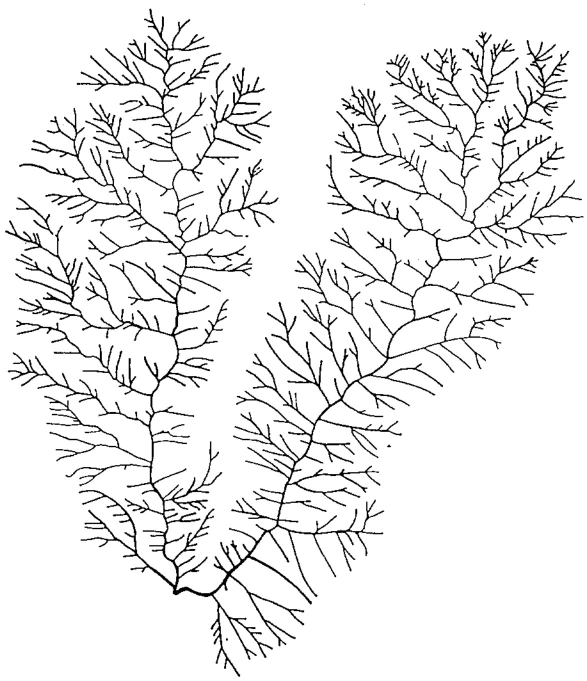

Fig. 3. The drainage network in the Volfe and Bell Canyons, San Gabriel Mountains, near Glendora, California obtained from field mapping [8].

4 a, with the number and mean lengths presented for streams of order 0 to 4. It is seen that the results correlate well with (1) taking $D = 1.81$. Shown in Fig. 4b are the applicable statistics for the entire United States [3], and the corresponding fractal dimension is $D = 1.83$. It is seen that stream networks are, to a good approximation, fractal and have fractal dimensions near 1.80, clearly somewhat less than the space filling $D = 2$. This is consistent with the example illustrated in Fig. 3. Two processes may be responsible for the network fractal dimension being less than 2. First, sparseness may be a property of the network itself, much as the fractal trees in Fig. 1 give varying fractal dimensions if the constructions are extended to infinite order. In addition, real networks are not truly space filling; there is always a characteristic smallest stream. Over short distances, small fluxes of water act diffusively, flowing over the surface or through the near-surface without incising channels (streams) [2]. Nevertheless, the fractal dimension given in (4) is derived from the entire spectrum of observed stream orders. Thus, the diffusional effects of short-range transport should only truncate the spectrum at the lowest order, not change its fractal dimension.

3. Previous models

A variety of models have been proposed to describe the statistics and origins of drainage networks [10]. Descriptive models were introduced by Shreve [11,12] and Scheidegger [13], in which drainage networks were considered as infinite topologically random networks (i.e., no one distribution of network links preferred over any other). They showed that the statistics of real drainage networks matched the most probable number–order distribution of a topologically ran-

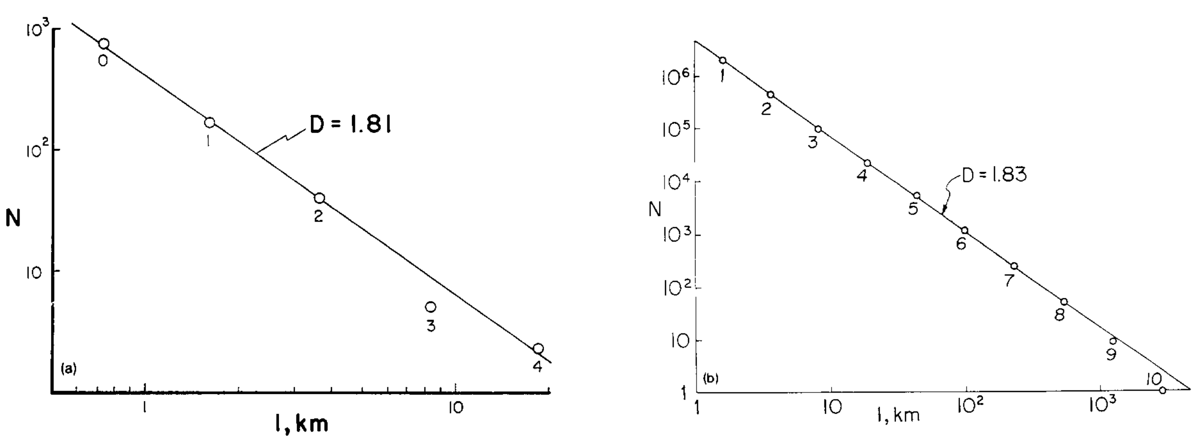

Fig. 4. Dependence of the number of streams of various orders on their mean length for (a) the example illustrated in Fig. 2 and (b) for the entire United States [2].

dom network. Although these models have proven useful as a way of describing fluvial networks, they contain little information on the dynamical processes that form them.

Other workers have proposed random growth models to explain the planform organization of drainage networks. Leopold and Langbein [14] and Shenck [15] proposed models in which the streams themselves followed random walks. Thus the network was not headward growing, but propagated laterally from the most central 'trunk' stream. In addition, the network grew by the addition of entire stream segments, rather than by gradual expansion (accretion). Howard [16] and Stark [17] introduced accretionary headward growth models. In Howard's [16] model, a site adjacent to the existing network was chosen randomly and the network propagated to this site. Thus, all sites on the network had an equal probability for growth. Stark [17] used an invasion percolation technique, in which the growing network was superposed on a fixed random field (analogous to a substrate with variable erodibility). At each time step, the network propagated to the adjacent site having the highest erodiblity value over the entire perimeter. Although all sites on the network had differing probabilities for growth, these probabilities did not change through time since the random field was fixed from the start.

A number of workers have coupled the evolution of drainage networks to the three-dimensional evolution of topography. Culling [18] initiated this approach, proposing that the downslope flux of eroded sediment is proportional to the topographic slope. Accordingly, the evolution of topography through time satisfies the diffusion equation. While solutions to the diffusion equation give satisfactory results for small-scale features (e.g., fault scarps and alluvial fans), diffusion cannot yield self-similar landforms [19]. Thus, diffusion alone is not a satisfactory model for the evolution of topography. Chase [20,21] has combined a cellular-automata advection model with diffusion and has generated reasonably realistic topography with drainage networks. Meakin et al. [22] have applied a DLA approach and Willgoose et al. [23] an advective–diffusive model. Kramer and Marder [24] have modeled the development of drainage networks on a water-covered landscape assuming that the erosion is proportional to the product of velocity and pressure.

4. DLA model

The model presented here is a modified version of the classical diffusion-limited aggregation (DLA) technique pioneered by Witten and Sander [25]. In general, DLA models generate fractal networks through iteration of random walks originating at a fixed distance from the existing network. Whenever one of the random trajectories intersects the network an element is accreted, thus producing network growth. The method has proven particularly useful in studies of immiscible fluid flow, in which the replacement of a high-viscosity fluid by a low-viscosity fluid results in dendritic fingering of the interface [26]. Because the random walkers introduced at the boundaries are shielded from the interior of the network as it expands, the network is sparse and the fractal dimension is low.

In our model the random walkers are introduced randomly over the entire grid and are allowed to walk until they either intersect the evolving network or are lost from the grid. We view the random walkers as unit water fluxes (rainfall and overland flow) which migrate over a relatively flat surface until they find a gully (network) in which to flow. When the flux joins the gully the latter erodes and expands the network. For spatially uniform 'rainfall' (i.e., initial placement of the walker), the probability for accretion at any given site on the network decreases with the degree of shielding near the site. Since this shielding changes through time as the network evolves, the probability for accretion at any site also changes through time. This relationship is not present in the planform models discussed above.

Kondoh and Matsushita [27] used a more classical DLA technique to model drainage formation in which random walkers were introduced at a fixed radial distance from a single seed cell. We feel that if precipitation is uniformly distributed over real landscapes, random walkers should be uniformly distributed over the entire grid. In addition, by using a single seed cell, Kondoh and Matsushita [27] neglect the inherent competition between adjacent subnetworks that occurs in real landscapes.

The exact mechanics of our model are illustrated in Fig. 5. A square grid of 15 × 15 cells is used in this illustration. Five seed cells are introduced at random points on the lower boundary. The evolving network must grow from these seed cells. For the example shown, sixteen cells have been accreted to the seed cells. Cells are allowed to accrete if one (and only one) of the four nearest neighbor cells is part of the pre-existing network. Prohibited sites which already have two neighboring sites occupied are identified by stars. Sites available for accretion to the network are indicated by circles. A random walker is introduced at a random cell on the grid and the hypothetical path is traced by the solid line. After 28 random walks it accretes to the network at the hatched cell. A random walk proceeds until either (1) the walker accretes to the network, (2) exits the grid, or (3) lands on a prohibited cell. In each case the walk is terminated and a new walker is introduced on a new, randomly selected site. The iteration of this basic procedure results in a branching network composed of linked drainage cells. This 'self-avoiding' algorithm prevents local clumping of drainage cells, and is similar to that used by Stark [17].

Although the model is highly schematic, the mechanics outlined here are analogous to the mechanics operating in real drainage systems. The accretionary nature of network growth produces a headward evolving drainage pattern similar to patterns of headward erosion seen in nature [28]. The accretion process itself is analogous to a flux of water (e.g., overland flow) intersecting an existing drainage, thereby initiating a new first-order channel. The self-avoiding algorithm prevents drainages from becoming locally space filling at the finest scales. In nature, this limit may be controlled by a threshold transition from diffusive slope processes to advective channelization processes [1]. The linear-stability analysis carried out by Loewenherz [29] also argues for a finite spacing between channels at the finest scales.

5. Results

Our simulations have been carried out on a 256 × 256 grid of cells with one-third of the bottom row tagged as seed cells. An evolutionary sequence for our drainage model is given in Fig. 6; the grey-scale is defined by Strahler's ordering system. Competition among the subnetworks results in the dominance of a few major stream systems. Furthermore, the major part of this self organization takes place fairly quickly, relative to the total number of iterations possible. By 30,000 iterations, the maximum order of the system ($N = 7$) has been established; subsequent events simply fill in the grid with lower order segments.

The networks shown in Fig. 6 bear a striking qualitative resemblance to the actual drainage

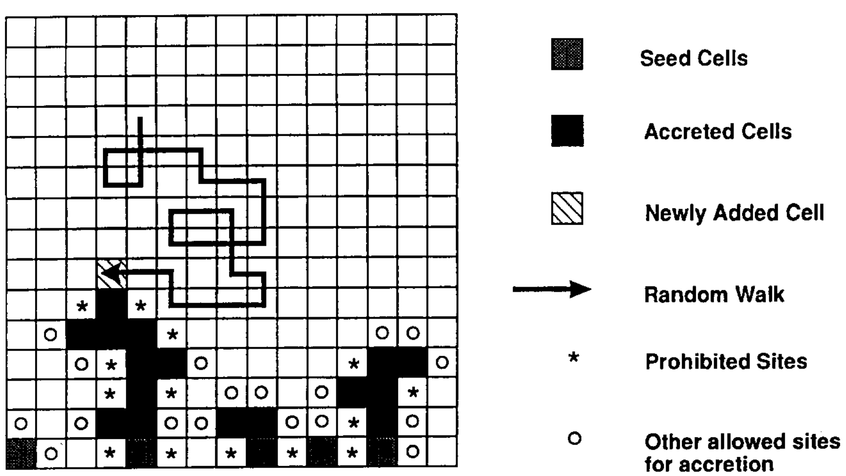

Fig. 5. Illustration of the mechanism for network growth in our DLA model. A random walker is randomly introduced to an unoccupied cell. The random walk proceeds until a cell is encountered with one (and only one) of the four nearest neighbors occupied (hatched). The new cell is accreted to the drainage network.

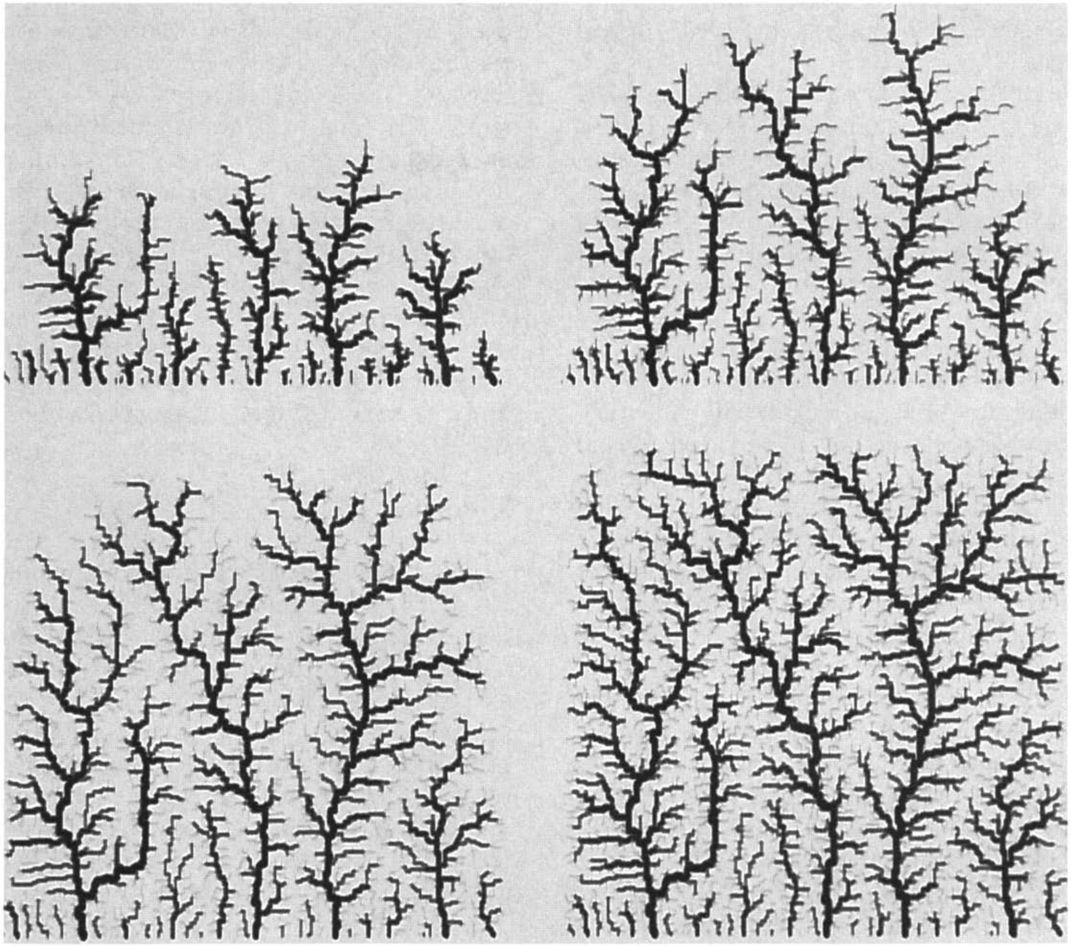

Fig. 6. An evolutionary sequence showing the development of the drainage network after (from upper left) 10,000, 20,000, 40,000, and 60,000 iterations. Each image is grey-scaled with, darker shading representing higher Strahler orders.

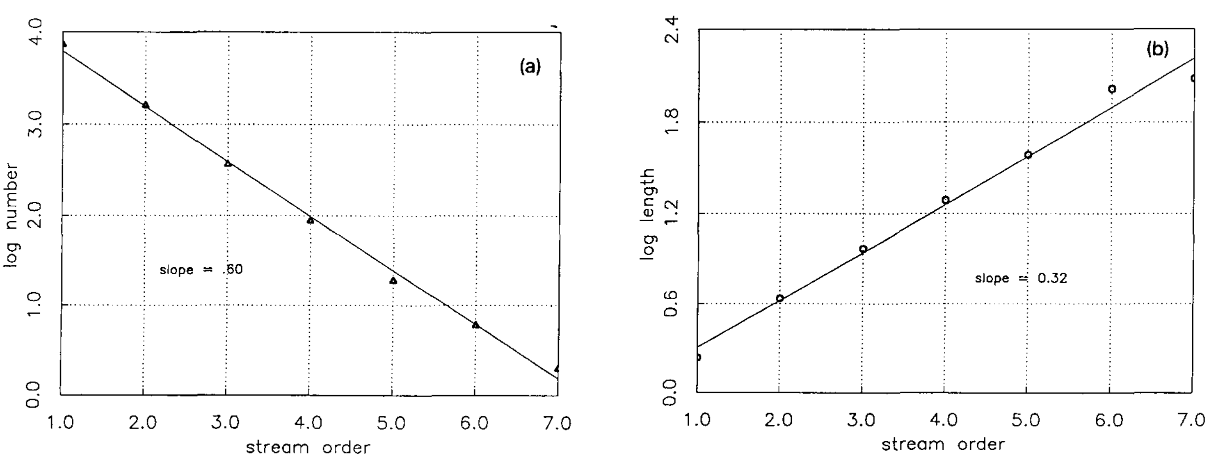

Fig. 7. Statistical results from our model. (a) The log of the number of stream segments of a given order plotted against that order. (b) The log of the average length of stream segments of a given order plotted against that order. Horton's laws state that a natural drainage basin will yield a linear relation on each graph.

network illustrated in Fig. 3. Seeing as the probability of accretion at a site increases with the amount of adjacent 'open' (unshielded) space, the greatest probability for growth is always at the upstream edge of the network. Thus, like real networks, the synthetic networks grow headward through time and are naturally elongate. The elongation is aided by the competition between adjacent subnetworks. The dependencies of number and mean length on order are given in Fig. 7 for our DLA model after 50,000 iterations. Both results are in excellent agreement with Horton's laws, i.e. a constant value of the bifurcation ratio R_b as defined in (2) and a constant value of the length–order ratio R_r as defined in (3). Using slopes derived from least-squares fits, we find $R_b = 3.98$ and $R_r = 2.09$. The corresponding value of the fractal dimension from (4) is $D = 1.87$. This is in good agreement with the values given in Fig. 4.

As with real drainages, we believe that the fractal dimension less than 2.0 reflects the underlying sparseness of the network at all orders, rather than the effects of the self-avoiding algorithm. Indeed, even if the self-avoiding algorithm were removed and the entire matrix was filled with drainage elements, the fractal dimension would not necessarily equal 2.0, since each cell of the matrix is considered to represent a unit length, not area.

6. Conclusions

The results of the modeling suggest that drainage networks represent a self-organized state, the formation of which may be described by a modified DLA technique. Specifically, the DLA technique produces planform drainage networks that bear a strong resemblance to real networks, evolve headward through time, and yield a 'sparse' fractal dimension of $D = 1.87$, slightly less than the space filling criterion of $D = 2$. This sparseness may reflect the dynamical interaction between network growth and shielding that is incorporated into the DLA model.

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