Calculation of Cell Areas on an Oblate Planet

**Purpose:** To calculate the cell areas on an oblate spheroidal planet. A “cell” is defined as a region bounded by two meridians of longitude and two meridians of latitude. It is approximately a rectangular patch (except for cells near the two poles, which are spherical triangles). To better approximation, a cell is a trapezoid. Here, we rigorously derive the area.

We define \( f \) as the flattening of the sphere,

\[
f = \frac{a - b}{a}
\]  

where \( a \) and \( b \) are the equatorial and polar radii, respectively. Note \( a > b \) for an oblate spheroid. For Earth, \( f = 0.00335364 \); for Mars, \( f = 0.00647630 \).

We begin with the definition of surface area in polar coordinates,

\[
\text{Area} = \int_{\lambda_1}^{\lambda_2} \int_{\beta_1}^{\beta_2} r^2 \cos \beta \, d\beta \, d\lambda
\]

where \( \lambda \) and \( \beta \) represent the longitude and latitude, respectively. For the sake of compactness, we do not carry the limits on the integrals in subsequent equations. However, the definite integrals remain to be evaluated after the dust settles.

The figure of an oblate spheroid is given (to order \( f^2 \)) by,

\[
r = a \left( 1 - f \sin^2 \beta - \frac{3}{8} f^2 \sin^2 2\beta \right)
\]  

Substituting (3) into (2) and doing the integration in the \( \lambda \) direction gives:

\[
\text{Area} = a^2 \Delta \lambda \int \cos \beta \left( 1 - f \sin^2 \beta - \frac{3}{8} f^2 \sin^2 2\beta \right) \left( 1 - f \sin^2 \beta - \frac{3}{8} f^2 \sin^2 2\beta \right) d\beta
\]

\[
= a^2 \Delta \lambda \int \cos \beta \left[ 1 - f \sin^2 \beta - \frac{3}{8} f^2 \sin^2 2\beta + f^2 \sin^4 \beta + \frac{3}{8} f^3 \sin^2 2\beta \sin^2 2\beta \\
- \frac{3}{8} f^2 \sin^2 2\beta + \frac{3}{8} f^3 \sin^2 2\beta \sin^2 2\beta + \frac{9}{64} f^4 \sin^4 2\beta \right] d\beta
\]

\[
= a^2 \Delta \lambda \int \left[ \cos \beta - f \sin^2 \beta \cos \beta - \frac{3}{8} f^2 \sin^2 2\beta \cos \beta + f^2 \sin^4 \beta \cos \beta + \frac{3}{8} f^3 \sin^2 2\beta \sin^2 2\beta \cos \beta \\
- \frac{3}{8} f^2 \sin^2 2\beta \cos \beta + \frac{3}{8} f^3 \sin^2 2\beta \sin^2 2\beta \cos \beta + \frac{9}{64} f^4 \sin^4 2\beta \cos \beta \right] d\beta
\]

\[
= a^2 \Delta \lambda \int \left[ \cos \beta - f \sin^2 \beta \cos \beta - \frac{3}{4} f^2 \sin^2 2\beta \cos \beta + f^2 \sin^4 \beta \cos \beta + \frac{3}{4} f^3 \sin^2 2\beta \sin^2 2\beta \cos \beta \\
+ \frac{9}{64} f^4 \sin^4 2\beta \cos \beta \right] d\beta
\]
\[ \Delta = a^2 \Delta \left[ \sin \beta - \frac{f}{3} \sin^3 \beta - \frac{3}{4} f^2 \int \sin^2 2\beta \cos \beta \, d\beta + \frac{f^2}{5} \sin^5 \beta \right. \]
\[ \left. + \frac{3}{4} f^3 \int \sin^2 \beta \sin^2 2\beta \cos \beta \, d\beta + \frac{9}{64} f^4 \int \sin^4 2\beta \cos \beta \, d\beta \right] \quad (8) \]

\[ = a^2 \Delta \left[ \left( \sin \beta - \frac{f}{3} \sin^3 \beta + \frac{f^2}{5} \sin^5 \beta \right) \right]_1 \left[ \left. - \frac{3}{4} f^2 \int \sin^2 2\beta \cos \beta \, d\beta \right]_1 \]
\[ + \frac{3}{4} f^3 \left. \int \sin^2 \beta \sin^2 2\beta \cos \beta \, d\beta \right|_1 \left. + \frac{9}{64} f^4 \int \sin^4 2\beta \cos \beta \, d\beta \right|_1 \quad (9) \]

To do the integrals in terms II–IV, note the following trigonometric identities:
\[ \sin 2\theta = 2 \sin \theta \cos \theta \]
\[ \sin^2 2\theta = 4 \cos^2 \theta \sin^2 \theta = 4(\sin^2 \theta - \sin^4 \theta) \] \quad (10a – 10c)
\[ \sin^4 2\theta = 16(\cos^4 \theta - 2 \cos^6 \theta + \cos^8 \theta) = 16(\sin^4 \theta - 2 \sin^6 \theta + \sin^8 \theta) \]

Term I is self-evident. Examining terms II–IV in turn,

**II:**
\[ -\frac{3}{4} f^2 \int \left( 16 \sin^4 - 2 \sin^6 \beta + \sin^8 \beta \right) \cos \beta \, d\beta \]
\[ = -\frac{3}{4} f^2 \left[ \frac{16}{5} \sin^5 \beta - \frac{2}{7} \sin^7 \beta + \frac{1}{9} \sin^9 \beta \right] \]
\[ = -\frac{12}{5} f^2 \sin^5 \beta + \frac{3}{14} f^2 \sin^7 \beta - \frac{1}{12} f^2 \sin^9 \beta \mid_{\beta_1} \quad (11) \]

**III:**
\[ \frac{3}{4} f^3 \int \sin^2 \beta \times 16 \left( \sin^4 \beta - 2 \sin^6 \beta + \sin^8 \beta \right) \cos \beta \, d\beta \]
\[ = 12 f^3 \int \left( \sin^6 \beta - 2 \sin^8 \beta + \sin^{10} \beta \right) \cos \beta \, d\beta \]
\[ = 12 f^3 \left[ \frac{1}{7} \sin^7 \beta - \frac{2}{9} \sin^9 \beta + \frac{1}{11} \sin^{11} \beta \right] \]
\[ = \frac{12}{7} f^3 \sin^7 \beta - \frac{24}{9} \sin^9 \beta + \frac{12}{11} f^3 \sin^{11} \beta \mid_{\beta_1} \quad (12) \]

**IV:**
\[ \frac{9}{64} f^4 \int 16 \left( \sin^4 \beta - 2 \sin^6 \beta + \sin^8 \beta \right) \cos \beta \, d\beta \]
\[ = \frac{9}{4} f^4 \int \sin^4 \beta \cos \beta \, d\beta - \frac{9}{2} f^4 \int \sin^6 \beta \cos \beta \, d\beta + \frac{9}{64} f^4 \int \sin^8 \beta \cos \beta \, d\beta \]
\[ = \frac{9}{4} f^4 \sin^5 \beta - \frac{9}{2} f^4 \sin^7 \beta + \frac{9}{64} f^4 \sin^9 \beta \]
\[ = \frac{9}{20} f^4 \sin^5 \beta - \frac{9}{14} f^4 \sin^7 \beta + \frac{1}{6} f^4 \sin^9 \beta \mid_{\beta_1} \quad (13) \]
Putting it all together, Eq. (9) becomes:

\[
\text{Area} = a^2 \Delta \lambda \left[ I + II + III + IV \right]_{\beta_1}^{\beta_2}
\]  

(14)

where \( \beta_1 \) and \( \beta_2 \) indicate the latitude limits to the region of integration.

Now time for the sanity checks:
- First, in the case of a sphere, \( f = 0 \), and all terms disappear except the lead term in \( I \):

\[
\text{Area}_{\text{sphere}} = a^2 \Delta \lambda \sin \beta |_{\beta_1}^{\beta_2}
\]

- In this case, and for \( \beta_2 = \pi/2 \) and \( \beta_1 = -\pi/2 \) and \( \Delta \lambda = 2\pi \), we wind up with the area of a sphere, \( 4\pi a^2 \).
- Also, in the limiting case of one of the poles, this expression reduces to \( a^2 \Delta \lambda \Delta \beta / 2 \), which is the expected area of the little triangle.
- Second, in the limit of \( \Delta \lambda \to 0 \) and/or \( \Delta \beta \to 0 \), \( \text{Area} \to 0 \).
- Third, all terms in the final expression are odd, indicating symmetry around the equator, as expected.

### Table of Sample Areas, in [km²]

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<th>latitude [deg]</th>
<th>spheroid</th>
<th>sphere</th>
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</table>
The following IDL procedure evaluates our expression for the case of Mars (a = 3397 km):

```idl
pro SPHEROID_AREA
radius=3397.d0 ; equatorial, in km
f=0.00647630 ; flattening
gs=0.5d0 ; gridsize, in degrees
halfgs=gs/2.d0
outfile='spheroid_area.dat'
openw,unit, outfile,/get lun
print,unit,'latitude Area(km^2,sphere) Area(km^2,spheroid)'
for lat=-90-halfgs,-90+halfgs,-gs do begin
  top=lat+halfgs & bot=lat-halfgs
  area=ldtor*gs*radius^2*(dotrig(top,gs)-dotrig(bot,gs))
  sphere_area=ldtor*gs*radius^2*(sin(ldtor*top)-sin(ldtor*bot))
  print,unit,lat,area,sphere_area,format='(1x,f6.2,2(3x,f12.6))'
endfor
close,unit
print, 'Done. Data written to ',strm(outfile)
DONE:
stop
end
```

function dotrig, lat, f, gs
f2=f*f & f3=f2*f & f4=f3*f
sinlat=sin(lat!ldtor) & sinlat2=sinlat*sinlat
sinlat3=sinlat^3 & sinlat5=sinlat^5
sinlat7=sinlat^7 & sinlat9=sinlat^9
sinlat11=sinlat^11
term1=sinlat-(f/3.d0)*sinlat3+(f2/5.d0)*sinlat5
term2=-(12.d0/5.d0)*f2*sinlat5+(13.d0/14.d0)*f2*sinlat7-(f2/12.d0)*sinlat9
term3=(12.d0/7.d0)*f3*sinlat7-(8.d0/3.d0)*f3*sinlat9+(12.d0/11.d0)*f3*sinlat11
term4=(9.d0/20.d0)*f4*sinlat5-(9.d0/14.d0)*f4*sinlat7+(f4/64.d0)*sinlat9
print,term1,term2,term3,term4
return,(term1+term2+term3+term4)
end