1. Theory of Radiative Transfer

"An expert is someone who has made all the mistakes that can be made in a narrow field." - *Niels Bohr*

Introduction

Consider an atmosphere. To first order this diffuse gaseous layer responds to heating from above by the Sun, e.g. if is Earth's, and/or by heating from below, if it is owned by a star or a lone brown dwarf, or by both, e.g. if it is owned by Jupiter. This heat establishes the temperature of the atmosphere and the equilibrium molecular forms of the atoms. In addition, the insolation drives certain disequilibrium processes that alter the chemistry from equilibrium, e.g. photochemistry or life. If a planet is differentially heated across it's globe, the partitioning of solar radiation, coupled with the planet's rotation, and the atmosphere's extent will set up the atmosphere's circulation. The solar energy can be further partitioned in a number of other ways, for example, into convection, and, if there is a condensible, latent heat and, if there is an ocean, ocean currents. One can imagine immediately that to understand the chemistry, temperature structure, circulation, cloud formation, ocean currents, and local and global climate of an atmosphere, one needs to know how incoming radiation is partitioned. This is radiative transfer.

Note that to fully understand Earth's climate, we would need to know, also, biology, geology, and oceanography. To understand an exoplanet's atmosphere, some studies of the interior structure and planetary evolution might be useful. But nonetheless, a good beginning is the energetics of the atmosphere.

Here we'll focus on the radiative transfer of the lower atmosphere, which includes tropospheres and stratospheres, but not the upper atmosphere, which includes the thermosphere, ionosphere and magnetosphere. We'll find that quite a detailed understanding of radiative transfer is needed to fully tackle most questions that we might need to face, either the understand the structure of an atmosphere or to determine the composition of a far away extrasolar planet.

Consider the fate of a photon in Earth's atmosphere as described by Figure 1. It can be absorbed or scattered by atmospheric gasses or particles, or the surface, or the ocean. It's energy can be converted to heat or evaporation, which would alter the temperature and composition of the atmosphere and therefore its absorption and scattering properties. So how should be begin to quantify this?

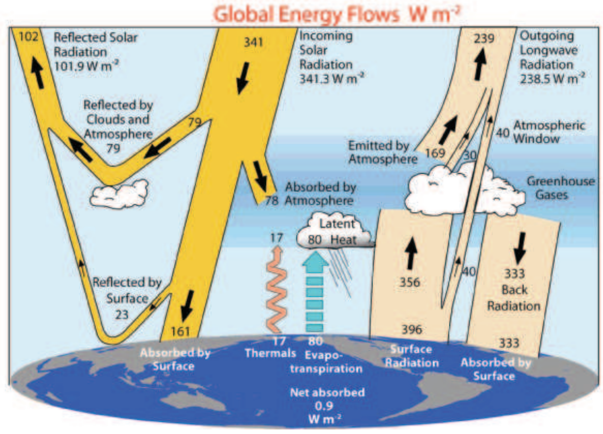

Fig. 1.— **Energy flow in Earth's atmosphere.**

A Few Definitions

First let's define some terms that can help us organize our thoughts. We must work with bunches of photons, since one visiting photon to our atmosphere does not effect the course of events too dramatically. For this reason we define several terms that quantify photon beams. But beforehand, we characterize the geometry of a beam, with the definition of a solid angle, a measure of angular area:

$$\Omega = \int\int_S \frac{\hat{\mathbf{r}} \cdot d\hat{\mathbf{a}}}{r^2}, \qquad (17)$$

Here, Ω is the solid angle subtended by an object of area $\mathbf{a}$, in the direction $\hat{r}$ (therefore the dot product) a distance r away. The bold symbols adorned with hats are unit vectors. The solid angle is the area of the section of a sphere of radius 1, delineated by e.g. a conic section. Alternatively one can visual the solid angle as a projection of an object on a spherical surface of 1 unit radius. An infinitesimal solid angle is:

$$d\Omega = sin(\theta) d\theta d\phi$$

and the largest possible value of a solid angle is:

$$\int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} sin(\theta) d\theta d\phi = 4\pi$$

Although the solid angle is unitless, we denote the unitless unit the steradian. Thus we designate the largest value of a solid angle as 4π steradians.

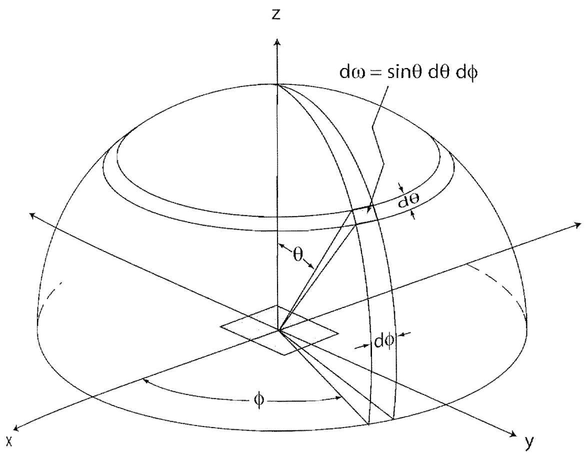

Fig. 2.— **A infinitesimal solid angle, $d\Omega$, here $d\omega$, is defined by $d\theta$ and $d\phi = sin(\theta)d\theta$**

Now we can begin to think of quantifying the energy transported by photons or equivalently an electro-magnetic field. There are two terms to define: the intensity (a scalar) and the flux (a vector).

The *intensity*, I_ν, also called the spectral radiance, quantifies the energy per unit area, wavelength, solid angle and time. Formally put, the intensity is defined as the energy, dE, transported by radiation of wavelengths λ to $\lambda + \delta\lambda$ (or frequency ν) across an element of area dA centered at some point P, and into a solid angle $d\Omega$, in the direction normal to the surface area, defined by the unit vector, $\hat{\mathbf{n}}$,

$$I_\lambda(P, \hat{\mathbf{n}}) = dE_\nu / \; dA \; d\Omega \; d\lambda \; dt, \quad (17)$$

or

$$I_\nu(P, \hat{\mathbf{n}}) = dE_\nu / \; dA \; d\Omega \; d\nu \; dt, \quad (17)$$

or, alternatively,

$$dE_\lambda = I_\nu(P, \hat{\mathbf{n}}) \; dA \; d\Omega \; d\lambda \; dt.$$

or

$$dE_\nu = I_\nu(P, \hat{\mathbf{n}}) \; dA \; d\Omega \; d\nu \; dt.$$

The dimensions of $I(P, s)$ are: ergs cm^{-2} sec^{-1} μm^{-1} sr^{-1}. The intensity thus does not depend on the orientation of any particular surface; it is defined only in terms of the direction of the beam. The intensity characterizes the emission of black bodies and car headlights.

However the intensity is not a measure of, e.g., the the insolation that heats the surface of a planet. The total energy flowing in a particular direction, e.g. $\hat{\mathbf{r}}$, is quantified by the *flux*, $\mathbf{F}_\nu$, a vector in the $\hat{\mathbf{r}}$ direction with a magnitude of:

$$F_\nu(P, \hat{\mathbf{r}}) = \int I(P, \hat{\mathbf{n}}) \; (\hat{\mathbf{r}} \cdot \hat{\mathbf{n}}) \; d\Omega, \quad (18)$$

where $\hat{\mathbf{r}} \cdot \hat{\mathbf{n}}$=cos($\theta$) where θ is the angle between vector $\hat{\mathbf{r}}$ and the vector $\hat{\mathbf{n}}$, the direction of the beam of the intensity. We have summed over all solid angles (Ω), therefore units of flux are: ergs cm^{-2} sec^{-1} μm^{-1} or Watts m^{-2} μm^{-1}.

Suppose that the Sun subtends a solid angle of $d\Omega_s$ and has a disk-averaged intensity of $\bar{I}_s$, its flux at Earth would then be equal to $\bar{I}_s \; d\Omega_s$ in the direction away from the Sun. Note that $\mathbf{F}_\nu \cdot \mathbf{S}$ (where $\mathbf{S}$ a surface) is the energy per unit time and frequency interval across the surface.

Note that the intensity can be visualized as equal to the component of the flux dF cos(Ω) *in the direction of the beam* within an infinitesimal solid angle $d\Omega$, such as that subtended by an astronomical object, divided by solid angle: $I \sim dF/dF\cos(\Omega)$.

The vector $\mathbf{F}_\nu$ can be written in terms of its components in three orthogonal directions, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, as the vector sum:

$$\mathbf{F} = \hat{\mathbf{x}} F_\nu(P, \hat{\mathbf{x}}) + \hat{\mathbf{y}} F_\nu(P, \hat{\mathbf{y}}) + \hat{\mathbf{z}} F_\nu(P, \hat{\mathbf{z}}).$$

However, usually we work with spherical coordinates with $\mathbf{z}$ in the upward direction.

Often it is useful to define two half range fluxes: the flux $\mathbf{F}^+$ that flows over the positive hemisphere, i.e. in the upward direction, and the flux $\mathbf{F}^-$ that flows over the negative hemisphere, the downward direction. Then the upward flux would be:

$$\mathbf{F}^+ = \int_+ I(\Omega) \; \hat{\mathbf{\Omega}} \cdot \hat{\mathbf{n}} \; d\Omega,$$

where $\hat{\mathbf{\Omega}}$ is the direction of the intensity beam and $\hat{\mathbf{n}}$ is the upward direction, normal to the surface.

These are, in spherical coordinates:

$$\mathbf{F}^{+} = \int_{+} I \cos(\theta) sin(\theta) d\theta d\phi \quad (19)$$

and

$$\mathbf{F}^{-} = \int_{-} I \cos(\theta) sin(\theta) d\theta d\phi \quad (20)$$

The total flux is then $\mathbf{F} = \mathbf{F}^{+} + \mathbf{F}^{-}$. If the intensity is independent of both position and direction (isotropic and homogeneous) the total flux is everywhere zero in the medium.

Note further that the if the incident intensity is isotropic, then the downward flux is simply:

$$\mathbf{F}^{-} = I \ \pi,$$

a relationship that also applies to the intensity of light measured from a far source from a telescope. This can be seen by comparing the flux and intensity of an astronomical object directly overhead. The intensity of the Sun at the top of the atmosphere (TOA) of Earth is:

$$I_{TOA} = (\frac{R_{Sun}}{D})^2 \ I_{Sun},$$

where I_{Sun} is the Sun's intensity at its surface, and R_{Sun} and D are the Sun's radius and the Earth to Sun distance. Now the flux, in the downward direction, is:

$$\mathbf{F} = I \ d\Omega \ \hat{\mathbf{z}},$$

since $d\Omega$ is small and $\cos(\theta)$=1. The fraction of the sky (of 2π solid angle) occupied by the Sun is the same as the fraction of the area of a hemisphere of radius D occupied by an object of area πR^2_{Sun}.

$$\frac{d\Omega}{2\pi} = \frac{\pi R^2_{Sun}}{2\pi D^2},$$

and therefore,

$$\mathbf{F} = \pi \ (\frac{R_{Sun}}{D})^2 \ I_{Sun} \quad \hat{\mathbf{z}}.$$

Again the flux and intensity differ by a factor of π. This point seems at first a bit inconsequential. However, it is in fact amazing how much time is wasted because one person writes their data as an intensity (with the little steradian unit) and another person reads it as a flux. So, watch out for π.

Generally we are interested in the intensity of light averaged over a particular solid angle, for example the mean intensity of burning coals or the Sun or a gamma ray burster. If we integrate the intensity over a specified solid angle, and normalize it by the solid angle of a sphere (4π) we have the *mean intensity*:

$$J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega. \quad (21)$$

Note that if the intensity, I_ν, is isotropic, $J_\nu = I_\nu$.

Light can also be quantified in terms of the *energy density*. For a cylinder of length ds, a photon takes a time dt=ds/c to cross; the volume can then be written as $A \ cdt$. The the energy transported is:

$$dE_\nu = \mu(\Omega) \ d\Omega \ dA \ c \ dt \ d\nu,$$

where $\mu(\Omega)$ is the energy density per unit solid angle. Equating this to the expression of dE_ν in terms of intensity (eq. 17),

$$\mu(\Omega) = I_\nu / c.$$

The energy density is then

$$\mu = \int \mu(\Omega) d\Omega = \frac{4\pi}{c} J_\nu. \quad (22)$$

Equation of Transfer

Lambert's Law

In a general sense, radiation interacts with matter in three different ways: through *absorption*, *scattering* and *emission*. During absorption, radiative energy is transformed into internal or kinetic energy. Scattering involves the transfer of radiative to internal to radiative energy. In such a case there can be a change in the frequency (thus energy), direction, and polarization of the light. Emission converts internal or kinetic energy into radiative energy.

For practical purposes, consider deliniating the interaction between light and matter as either *extinction* or *emission*. The intensity of light decreases for an extinction process and increases during emission. The extinction process can be likened to a collision process in atomic physics where bombarding particles cross a cavity filled with target particles. Here the probability of collision increases with the total cross section of the target particles in the cavity, as well as with the number of bombarding particles. Similarly, the attenuation of radiation depends the density of gas or particles and the path length, in addition to the incident intensity. The decrease in intensity along a path ds is proportional to the amount of matter intercepted by the incident radiation:

$$dI_\nu = -(\kappa_a + \kappa_s)_\nu \ \rho \ I_\nu ds. \tag{23}$$

Here ρ is the mass density and thus κ_a is the mass absorption coefficient and κ_s is the mass scattering coeffiecient (units of $\mathrm{cm}^2/\mathrm{gm}$); their sum is the extinction coefficient, κ_ν. This expression is called Lambert's law in Germany (after Johann Lambert, a german mathematician) and Bouguer's law in France (after Pierre Bouguer, a french mathematician). Here in the US, you often find this under Beer's law (after August Beer, a german physicist, who worked 100 years after the latter two, and about whom there no evidence for the discovery of such an exponential expression). Note that we can also express the attenuation in the intensity in terms of the number density $n = \rho/m$ of the absorbers and scatters (each of mass m):

$$dI_\nu = -(\sigma_a + \sigma_s)_\nu \ n \ I_\nu ds,$$

where σ_a and σ_s are the absorption and scattering cross sections. Generally stimulated emission is also included in the absorption coefficient.

The emitted radiation is proportional to the amount of matter present:

$$dI_\nu = +j_\nu \ \rho ds, \tag{24}$$

where j_ν is the emission coefficient (units of erg gm^{-1} sec^{-1} sr^{-1} Hz^{-1}). The emission coefficient can also be defined in terms of volume (with units erg cm^{-3} sec^{-1} sr^{-1} Hz^{-1}), in which case $dI_\nu = j_\nu ds$. The *emissivity*, ϵ_ν, is the angularly integrated emission coefficient, and has units erg gm^{-1} s^{-1} Hz^{-1}. Note that the emission coefficient can in part be due to scattering of light back into the original beam.

Hence, considering both absorption and emission (eqs. 23 and 24), we can write down the expression for the change in intensity,

$$\frac{1}{\rho}\frac{dI_\nu}{ds} = -(\kappa_a + \kappa_s) I_\nu + j_\nu, \tag{25}$$

known as the equation of transfer. We can simplify the equation introducing the *source function* S_ν, defined as the ratio of the emission to the extinction coefficients:

$$S_\nu = \frac{j_\nu}{(\kappa_a + \kappa_s)}. \tag{26}$$

and the optical depth:

$$\tau = \int \kappa_\nu \rho ds.$$

Then,

$$\frac{dI_\nu}{d\tau} = -I_\nu + S_\nu. \tag{27}$$

When Boltzmann's law for the distribution of molecules in quantum states applies, the source function is the Planck's function. This law applies for systems where collisions govern the population of states. In the upper atmosphere, at a certain level emissions occur more quickly than collisional de-excitation. At these levels the distribution of states is not Boltzmannian and the Planck function must be derived according to the distribution of states.

2. The General Solution

The radiative transfer equation,

$$\frac{dI_\nu}{d\tau} = -I_\nu + S_\nu,$$

can be solved generally by multiplying both sides of the equation by e^{τ_ν}:

$$\frac{d[I_\nu e^{\tau_\nu}]}{d\tau_\nu} = e^{\tau_\nu} S_\nu.$$

Integration from point A to point B, an optical depth of τ_ν away from A, provides the following solution for I_ν:

$$I_\nu(\tau_\nu) = I_\nu(0) \ e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu e^{-(\tau_\nu - \tau'_\nu)} d\tau'. \tag{28}$$

The intensity at B (left hand side) equals the initial intensity at A attenuated by the optical depth between A and B (τ) plus the integrated source function (the radiating and scattering elements) along the path from A to B, attenuated by the intervening optical depth.

Generally one can not solve this equation analytically because the source function often includes the (multiple) scattering of light into the beam, which depends on the attenuation of light from scattering,

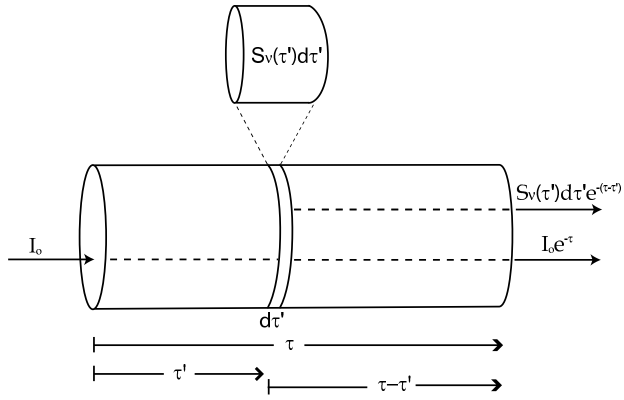

Fig. 3.— **The final intensity equals the original intensity attentuated by absorption and scattering events, plus contributions from the source function along the path (also attenuated by the remaining column of gas).**

which depends on the intensity of light. Scattering is often non-isotropic, thereby rendering complicated angular dependencies. Yet, we can solve this equation analytically if the source function is constant:

$$I_\nu(\tau_\nu) = I_\nu(0)\ e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}), \tag{29}$$

or,

$$I_\nu(\tau_\nu) = S_\nu + e^{-\tau_\nu}(I_\nu(0) - S_\nu).$$

We can see from this expression, that as $\tau_\nu \to \infty$ that $I_\nu \to S_\nu$. In optically thick media, the intensity approaches the source function. For this reason, radiative transfer models of the infrared radiation of jovian planets, specify the black body intensity at a deep atmospheric level as the boundary condition for the intensity at the bottom of the atmospheric model.

A Simple Solution: No source, no scattering

Assume that S_ν=0 and κ_s=0. The solution to the equation of radiative transfer is then simply

$$I_\nu(s) = I_\nu(0) e^{\int_0^s -\kappa_{a,\nu}\rho ds'}.$$

The optical depth of an atmosphere, defined as:

$$\tau = \int_0^s \kappa_\nu \rho ds'$$

is a measure of the amount of extinction due to scattering and absorption, which here is only absorption. Then:

$$I_\nu(s) = I_\nu(0)\ e^{-\tau}, \tag{30}$$

and the point s where τ =1, is the level in the atmosphere where the intensity is attenuated by $\frac{1}{e} = 0.37$. This simple expression is often referred to in the US as "Beer's Law", for, as we've noted, apparently no good reason.

3. Spectra of Exoplanets

We can actually do something with this simple solution to the radiative transfer equation! It's good for analyzing the occultation of stars or the Sun behind the limb of a planet's atmosphere. One recent study involves studies of the composition of atmospheres of planets that lie outside the solar system.

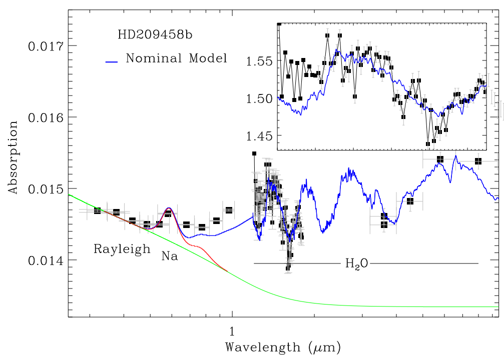

Fig. 4.— **Data of exoplanet HD209458b from several sources, and my RT calculations of the spectrum, such that the green model includes only Rayleigh scattering, the red model has Na emission as well, and the blue model includes water opacity.**

The composition and structure of exoplanetary atmospheres are probed with two kinds of spectra. During the primary eclipse, an exoplanet passes in front of its star, and the transmission of the star's light through the planet's atmosphere on the limb can be discerned. During the secondary eclipse, an exoplanet passes in behind of its star; the difference between the star plus planet and the star's spectra yields the emission spectrum of the planet. Here we look at calculations of planetary transmission spectra. Later, we will discuss calculations of planetary emission spectra.

Measurements of the transmission through an exoplanet's atmosophere, as well as occultations of planetary atmospheres do not probe the vertically integrated column abundance, but rather a tangentially integrated density, N_t. If we assume that there is no thermal emission, and that all the scattered light is scattered out of the beam then we can approximate the equation of radiative transfer by assuming no source function. The transmission through the atmosphere simply given by Beer's law.

With the radiative transfer equation solved, all the work goes into defining the tangentially integrated densities of the particulates and gaseous species and integrating this by the extinction and absorption coefficients to determine the optical depth, τ. This is usually done by dividing the tangential path into infinitesimal lengths, and calculating the column abundance of the discrete bits of atmosphere, which when summed provide the total tangential column abundance. The tangentially integrated density can also be estimated, particularly when the opacity source is well mixed in the atmosphere, because most of the contribution derives from the closest point of the tangent line, a distance R from the center of the planet. Here the density changes gradually along the tangent line.

Integrate the density along tangent line, s, an infinite distance in either direction,

$$N_t(R) = \int_{-\infty}^{\infty} N(r) ds,$$

where $r(s)$ is the distance to the center of the planet at a point along the tangent, and $N(r)$ is the atmospheric density at r. Change the dependent variable to β, the angle between R and r. Then:

$$r = R \ \sec\beta,$$

$$s = R \ \tan\beta,$$

and

$$ds = R \ \sec^2\beta \ d\beta.$$

We approximate:

$$N(r) = N(R) e^{-(r-R)/H}$$

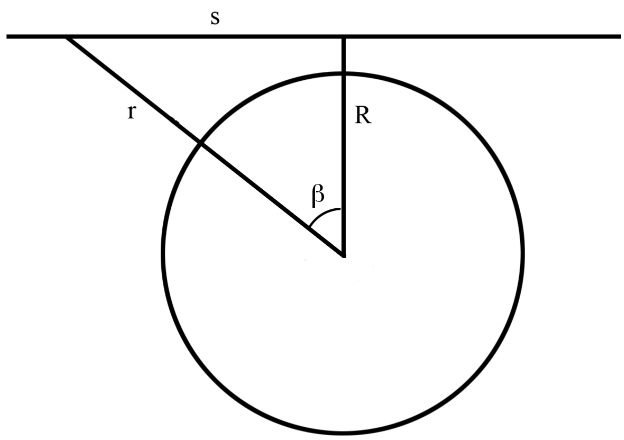

Fig. 5.— **Integration of the tangent column density**

where H is the atmospheric scale height:

$$H = \frac{R_g T}{mg},$$

where R_g is the gas constant, T the temperature, m the mean molecular weight, and g the gravity.

Then:

$$N_t(R) = \int_{-\pi/2}^{\pi/2} N(R) e^{-R(\sec\beta - 1)/H} \ R \ \sec^2\beta \ d\beta,$$

the solution of which is a modified Bessel function of the third kind, K_1:

$$N_t(R) = 2N(R) \ R \ e^{R/H} \ K_1(R/H)$$

or

$$N_t(R) = N(R) \ (2\pi RH)^{1/2} \ (1 + \frac{3}{8} H/R + ...).$$

In summary, the column density along the tangent cord can be approximated as:

$$N_t(R) = \int_{-\infty}^{\infty} N(r) ds \approx N(R) \ (2\pi RH)^{1/2}$$

The atmospheric transmission $T(R)$ of light through a cord that is a distance R from the planet's center is simply:

$$T(R) = e^{-N_t(R) \ \sum_i \kappa_i}$$

where we sum the extinction, κ over all sources of opacities (i.e. gases and particulates).

The dip of light caused by an exoplanet occulting its star is caused by the blockage of star light by the planet. Emission from the planet's atmosphere is negligible. Most papers in the field present their calculations as "Absorption" values, which is, for each wavelength, the ratio of effective area that the planet occults to the area of the star:

$$\text{Absorption} = \frac{\pi R_P^2}{\pi R_S^2} + \int_{R_P}^{\infty} 2\pi r(1 - T(R))dr/\pi R_S^2,$$

where R_P is the radius of the planet at 10 bars (where the planet's atmosphere is opaque), R_S is the primary star's radius, and $T(R)$ is the atmospheric transmission of light through a cord that is a distance R from the planet's center. The first term represents the occultation of the opaque part of the planet; the second term represents the occultation of the planet's atmosphere. The "Absorption" is thus the predicted variation in the observed light curve depth as a function of wavelength. Actually some of the second term on the right side in the equation above covers regions that are optically thick an optically thin. It could be written in terms of 3 terms:

$$\int_{R_P}^{\infty} 2\pi r(1-T(R))dr/\pi R_S^2 = \frac{R_O^2 - R_P^2}{R_S^2} + \int_{R_O}^{R_T} 2\pi r(1-T(R))dr/\pi R_S^2,$$

where R_O and R_T are the distances from the planets center, where to within a certain fraction, say 0.05, no light is transmitted through the limb and to within a 95% transmission no light is absorber. If these values are adopted, the transmission spectrum samples only 4 scale heights of the atmosphere. The only extrasolar planet for which we have both ample data and "interpretive" data is called HD209458b. This is a Jupiter-sized body, which revolves about a G star at a distance of 0.0475 AU every 3 and a bit days. It therefore quite hot (T_{eff} ~1200 K). Note that Mercury's semi-major axis is 0.39 AU. The brightness of the primary star (V=7.65) and the proximity of the planet, allows us to pick out the transmission of starlight through the planet's atmosphere, when the planet goes in front of the star. There is another bright system with a planet called HD189733b, which is featured in your interesting homework, for which we also have quite a bit of spectra. However these spectra, I think, are difficult to make sense of.