

1. Scattering RT Calculations

“We come spinning out of nothingness, scattering stars like dust.” - Jalal ad-Din Rumi (Persian Poet, 1207-1273)

We’ve considered solutions to the radiative transfer equation for the obvious situation in which there is no source function. We’ve also solved the radiative transfer for an isotropic source function, by making the common approximation that the atmosphere can be subdivided into plane parallel levels that are thin enough that the temperature and composition can be assumed to be constant within the layer. In this case the source function is not only isotropic but also constant within the layer.

Here we consider that case where the source function is neither constant, nor isotropic. This case applies to the scattering of sunlight off of planetary atmospheres, particularly for Titan, which has hazy atmosphere, with particles that are highly forward scattering. To address the non-constant nature of the source function, we again subdivide the atmosphere into homogeneous layers. To address the anisotropic nature of the source function we discretize the angular integration of the source function. However, it is important to note that there are other ways of solving the radiative transfer equation, that don’t involve any of these quantization tricks.

1.1. Phase Function

To quantify the scattering of particulates we define the phase function, $p(\theta', \phi', \theta, \phi)$, which depends on the direction of the incident light, characterized by θ' and ϕ' , and the direction of the outgoing radiation of direction θ and ϕ . As such the scattering characteristics of one oddly shaped particle depends on four variables, and can be quite complicated. However, in an atmosphere the suspended particles usually have no preferred orientation. In this case, there is no preferred orientation, and the scattering phase function depends only on the *scattering angle*, i.e. the angle (Θ) between the direction of incident and scattered radiation (Fig. 1). Exceptions to this rule are snow flakes, which become oriented, and raindrops, which become pie-shaped, as they fall.

The *phase function* is defined as the angular cross section per particle, $\sigma(\theta', \phi', \theta, \phi)$, normalized to the angular cross section integrated over a complete solid angle:

$$p(\theta', \phi', \theta, \phi) = \frac{\sigma(\theta', \phi', \theta, \phi)}{\int_{4\pi} d\Omega \sigma((\theta', \phi', \theta, \phi))/4\pi}.$$

or, assuming “disoriented” particles such that

$$p(\theta', \phi', \theta, \phi) = p(\cos \Theta),$$

we have:

$$p(\cos \Theta) = \frac{\sigma(\cos \Theta)}{\int_{4\pi} d\Omega \sigma(\cos \Theta)/4\pi}.$$

Thus:

$$\int_{4\pi} \frac{p(\cos \Theta)}{4\pi} d\Omega = 1 \quad (1)$$

or

$$\int_{-1}^1 \frac{p(\cos \Theta)}{2} d\cos\Theta = 1 \quad (2)$$

The probability of a scattering event in a region of solid angle of $d\Omega$ around Θ is

$$\frac{p(\cos \Theta)}{4\pi} d\Omega. \quad (3)$$

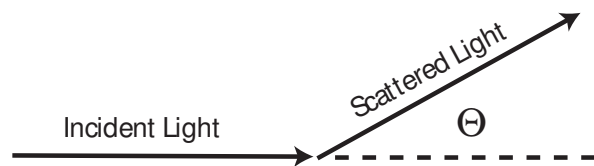


Fig. 1.— Definition of the scattering angle, Θ .

A particle’s phase function depends on it’s size relative to the wavelength of incident light, λ , which is defined by the *size parameter*, x :

$$x = \frac{2\pi r}{\lambda}, \quad (4)$$

where r is the size or radius. In addition the phase function depends on the particle’s shape. Fig. 2 shows how spherical particles display more forward scattering the larger the size parameter. If the size parameter is small enough then light is scattered equally in the forward and backward direction. This can be seen from *Rayleigh scattering* (named after Lord

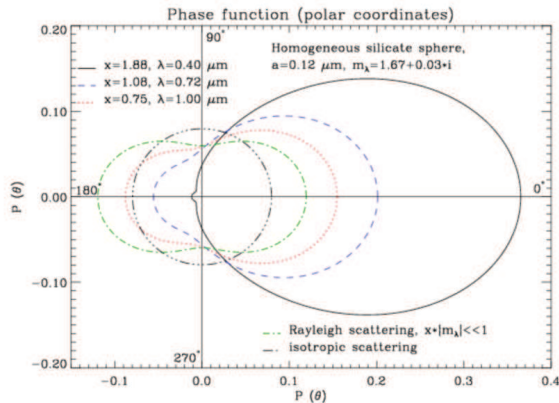


Fig. 2.— Phase functions for different x .

Rayleigh, who in 1871 worked out its properties), which is the scattering of particles for which $x \ll 1$. It characterizes therefore the scattering of sunlight by the molecules in an atmosphere.

1.2. Source Function for Scattering

From the definition of intensity of light, I_ν , we can write the energy *incident* normally on an area dA in the direction (θ', ϕ') , within the solid angle $d\Omega'$ in time dt and frequency interval $d\nu$ centered at ν is:

$$dE'_\nu = I_\nu(\theta', \phi') dA dt d\nu d\Omega'$$

The energy *scattered* is attenuated by $\sigma_s ds$ (where ds is the path length and σ_s is the scattering cross section).

Now $p(\theta', \phi'; \theta, \phi)/4\pi$ is essentially the probability density function, as evident from Eq. 1. Therefore the incoming intensity of light multiplied by this probability density tells us what fraction of light, coming from the direction (θ', ϕ') , is scattered into the solid angle $d\Omega$ in the direction (θ, ϕ) . The total scattering energy emerging in a direction (θ, ϕ) from a volume element $dV = ds dA$ from all the radiation from all incident solid angles can then be derived by integrating over the incoming incident angles:

$$dE_\nu = \sigma(\nu) dV dt d\nu d\Omega \int_{4\pi} d\Omega' \frac{p(\Theta)}{4\pi} I_\nu(\theta', \phi').$$

The emission coefficient for scattering is then:

$$j_\nu = \frac{dE_\nu}{dV dt d\nu d\Omega} = \sigma_s \int_{4\pi} d\Omega' \frac{p(\Theta)}{4\pi} I_\nu(\theta', \phi').$$

And the source function for scattering is:

$$S_\nu(\theta, \phi) = \frac{j_\nu}{\kappa_\nu} = \frac{\sigma_s}{\sigma_\nu} \int_{4\pi} d\Omega' \frac{p(\Theta)}{4\pi} I_\nu(\theta', \phi'), \quad (5)$$

where σ_ν is the total extinction coefficient, equal to the sum of the scattering and absorption cross sections, σ_s and σ_a . The term $\sigma_s/\sigma_\nu = \sigma_s/(\sigma_s + \sigma_a) = a_\nu$ is called the *single scattering albedo* and, less than unity, is the probability of a scattering for each extinction. The probability for absorption is then $1 - a_\nu$. The radiative transfer equation for multiple scattering and emission can be written as:

$$\begin{aligned} \frac{dI_\nu}{d\tau} &= -I_\nu + [1 - a_\nu] B_\nu(T) \\ &+ \frac{a_\nu}{4\pi} \int_{4\pi} d\Omega' p(\Theta) I_\nu(\theta', \phi'). \end{aligned} \quad (6)$$

We integrate the incident intensity (primes) over all solid angles to determine the contribution of scattered intensity into the particular solid angle of interest.

For many problems, the atmosphere can be treated as a collection of overlying parallel layers, which if subdivided finely enough, are of uniform composition and temperature. For such a *plane-parallel* atmosphere it is common to designate τ as the vertical optical depth such that the actual path of light through a layer depends on the zenith angle, θ , at which photons enter the layer. The optical depth of this slant path is then:

$$\tau_s = \tau/|\mu|, \quad (7)$$

where $|\mu| = |\cos \theta|$.

2. Particle Scattering

Generally gas and particles do not scatter isotropically. The phase function, scattering efficiency, and single scattering albedo depend on the size of the scatterers relative to the wavelength of light, the shape of the particles, and for large particles, the indices of refraction. Scattering can change the polarization of light, which then also must be considered in a full description of scattering effect. We will begin by defining a few terms and then examine a few particular cases of the single scattering of light.

3. Rayleigh Scattering

Rayleigh scattering describes the scattering of sunlight by gas molecules in the atmosphere, and was originally formulated by Lord Rayleigh (1871) to explain the color and polarization of the light from the sky. More generally, it describes the scattering of light by particles much smaller than the wavelength of the light, and smaller than the wavelength of the light divided by the magnitude, $|n|$, of the index of refraction ($n_r - in_i$). In fact, the scattering properties of light depend on the size to wavelength ratio. For this reason we define the *size parameter*, x as:

$$x = \frac{2\pi r}{\lambda}. \quad (5)$$

For Rayleigh scattering, $x|n| \ll \lambda$.

When such a small particle is exposed to E-M waves, every part of it experiences simultaneously the E-M field. As a result the particle becomes polarized, such that the negative charges are displaced from the more massive (and therefore more fixed) positive nucleus. As a result of the charge separation the particle acquires a dipole moment, \mathbf{p} , which oscillates because of the externally applied electric field¹. The particle then radiates as a result of the accelerating charges. An oscillating dipole produces an oscillating electric field, \mathbf{E}_{scat} , (and therefore an outward propagating EM wave), which is proportional to the frequency of the oscillation squared. Since the intensity, i.e. the emitted areal power, is proportional to the square of the amplitude of the E field, the resulting cross section to Rayleigh scattering depends on wavelength roughly as frequency to the fourth power, i.e. $\sigma_{\text{Ray}} \propto \lambda^{-4}$, which paints most sunsets (Fig. 2). The Rayleigh scattering optical depth for an atmosphere of a particular composition can be obtained from the following website: http://pds-atmospheres.nmsu.edu/education_and_outreach

For a particle of isotropic polarizability the induced dipole moment of the particle is in the same direction as the externally applied electric field. Similar to a classical dipole, the resulting radiation field, i.e. the electric field of the scattered light, has an amplitude, \mathbf{E} , that is proportional to the projection of the dipole moment in the direction of the observer. Thus the intensity of the scattered light, which is proportional to \mathbf{E}^2 is proportional to $\sin^2(\theta)$, where θ is the angle

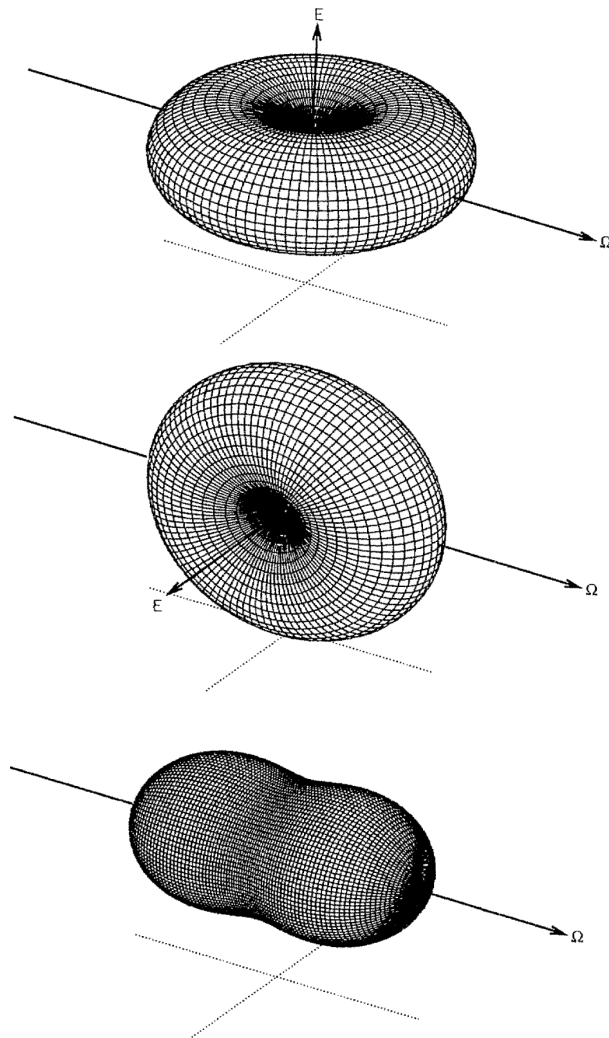


Fig. 3.— Rayleigh scattering of vertically polarized (top) and horizontally (middle) polarized light scatter polarized light in the different directions shown. Unpolarized light is made up the sum of equal amounts of the orthogonally polarized components. (From Petty)

¹A dipole moment equals the amount of the charge times the effective displacement distance of the charges.



Fig. 4.— Sunset. Ipanema, Rio de Janeiro.

between the direction of the dipole moment and the scattered radiation. Thus the phase function depends on the polarization state of the incident light. In addition, Rayleigh scattered light is, depending on the scattering angle, polarized (Fig. 1).

3.1. Polarization

Consider light moving in the \hat{x} direction, that is linearly polarized with an E field in the \hat{z} direction (Fig. 1, top). The dipole moment is then in the \hat{z} direction. The resulting scattered light is then polarized in the \hat{z} direction, and has the highest intensity in the \hat{y} direction (where $\theta=0$), and has zero intensity in the \hat{x} direction. Now for transverse EM waves, the E-field is perpendicular to the propagation direction. Therefore, natural unpolarized light can be considered as the sum of two linearly polarized waves (in the \hat{y} and \hat{z} directions) of equal intensity (Fig. 1, bottom). Therefore, for a single scattering event, unpolarized light scattered at $\Theta=90^\circ$ scattering angle is 100% polarized. In contrast, in the forward and backward directions ($\Theta=0^\circ$ and $\Theta=180^\circ$) light is unpolarized (Fig. 1). The degree of polarization is:

$$P = \frac{1 - \cos^2\Theta}{1 + \cos^2\Theta}. \quad (6)$$

However for multiple scattering events the the ambient light becomes unpolarized. In addition air molecules (i.e. N_2) are non-spherical and therefore

are not entirely of isotropic polarizability. Also scattering due to clouds and aerosols does not polarize light, and therefore diminishes the polarization of the sky. The phase function of Rayleigh scattering looks rather like a symmetric potato, and depends on the scattering angle as:

$$I_{Ray}(\cos \Theta) = \frac{3}{4} (1 + \cos^2(\Theta)). \quad (7)$$

3.2. Mie Scattering

If the particles are of similar size or a bit larger than the wavelength, the Rayleigh approximation can not be used. The scattering of spherical particles can be solved analytically, and is called *Mie Scattering* after Gustav Mie (18-69-1957) the first to publish the solution. There are numerous codes available to calculate Mie Scattering both using fortran, C or matlab; in addition, there is an online Mie Scattering Calculator at http://omlc.ogi.edu/calc/mie_calc.html and <http://www.lightscattering.de/MieCalc/eindex.html>.

We will not go into the derivation of the Mie Scattering solution, and instead examine the results of geometrical optics approximation of scattering off a single spherical particle, as discussed in Hansen and Travis (1974) and derived by Liou and Hansen (1971). They consider the paths of a number of parallel rays incident on the particle and calculate the reflection and transmission coefficients from Fresnel's equations and the direction of refracted rays from Snell's law. The scattered light from all incident rays is summed to derive the phase function, and polarization of the scattered light.

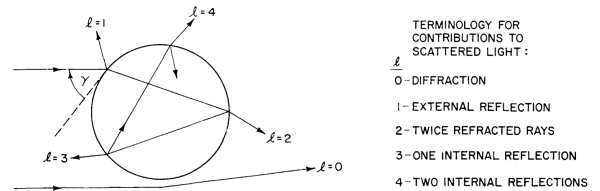


Fig. 5.— The paths of rays encountering a spherical particle. From Hansen and Travis (1974).

They subdivide the paths of the rays into categories that depend on the number of times, l , the ray encounters the boundary of the sphere. Rays that

entirely miss the particle ($l=0$) are diffracted in the forward direction (Fig. 4). The amount of diffracted light equals the amount striking the particle, independent of the particle shape and refractive index. Therefore the *scattering efficiency*², Q_{scat} , of large non-absorbing particles is 2. For non-absorbing particles the diffraction constitutes half of the scattered light. For fully absorbing particles the scattered light consists mainly of diffracted light, and Q_{scat} drops to nearly 1. The polarization of diffracted light mimics the incident light.

Rays that experience one external reflection ($l=1$) causes radiation scattered a 80-120 scattering angle to be positively polarized. However this component to the overall scattered light is smaller than that of refracted twice light ($l=2$) for transparent and partially transparent spheres. The refracted light concentrates in the forward direction, and is negatively polarized (Fig. 3). Light that is internally reflected ($l \geq 3$) represents only a small percent of the scattered light. The single reflection component ($l=3$) gives rise to a rainbow; the second reflection component ($l=4$) causes a secondary weaker arc. On Earth, rainbows are seen at an angle of 42° from the Sun. On Titan, since the raindrop is made out of methane, which has different indices of refraction from water, rainbows would appear at 49° .

3.3. Indices of refraction

The scattering behavior of particles depend not only on their size and shape but also on the *refractive index*, n , of the material:

$$n = n_r + in_i.$$

The real part of the index, n_r , represents the phase speed of the wave, while the imaginary part, n_i , represents the absorption of light propagating through the medium. More precisely:

$$n_r = \frac{c}{v},$$

where c is the velocity of light in a vacuum and v is the velocity in a medium. The change in the speed of light across a medium, effectively causes the light to bend. That is, the angle of the light ray to the normal of the interface between two media, θ , changes as light

² Q_{scat} is defined as the ratio of the scattering cross section, σ_{scat} to the geometrical cross section, i.e. πr^2 for a sphere.

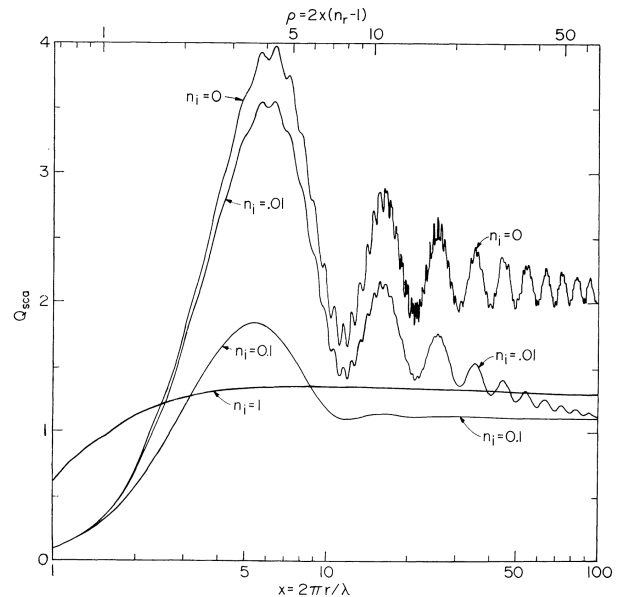


Fig. 6.— The scattering efficiency of a material of $n_r = 1.33$, like water as a function of size parameter and n_i . From Hansen and Travis (1974).

passes from material of one refraction index, n_{r1} to another, n_{r2} :

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} = \frac{n_{r2}}{n_{r1}},$$

where the angles are shown in Fig. 5

This called Snell's law in the USA after the Dutch astronomer Willebrord Snellius (1580-1626). It is called Descartes' law in France after the French mathematician René Descartes (1596-1650). You can imagine that the real index of refraction affects the refracted light within the particle, giving rise to a different angle between the Sun and a rainbow (Table 1). In addition it affects the phase function of the particle, through the destructive and constructive interference of light rays.

3.4. Asymmetry Factor

In general, forward scattering increases with particle size. This can be seen by calculating the asymmetry factor, g , which is the mean cosine of the scat-

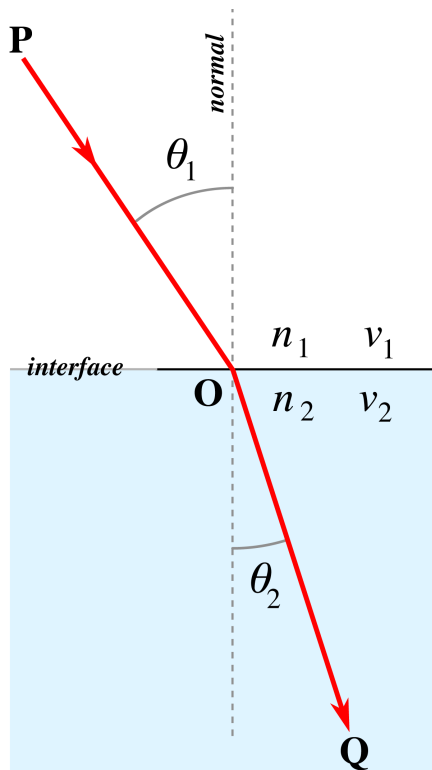


Fig. 7.— Light passing from a medium of index n_{r1} to index n_{r2} where $n_{r2} > n_{r1}$, since $\theta_2 > \theta_1$.

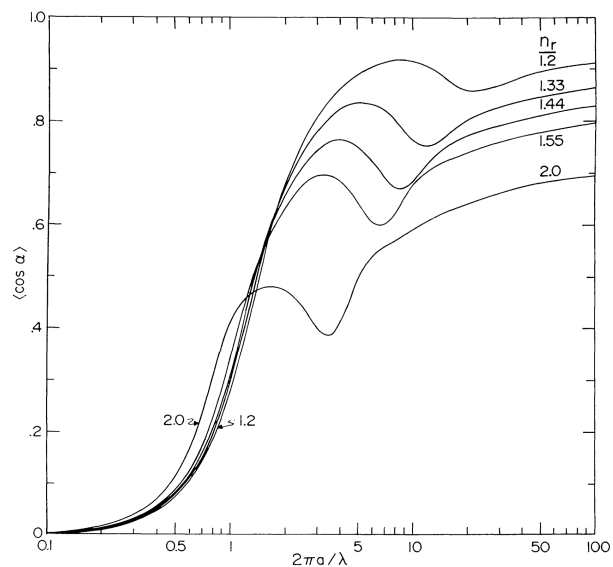


Fig. 8.— The asymmetry factor of a material of $n_r = 1.33$, like water as a function of size parameter and n_r . From Hansen and Travis (1974).

tering angle, weighted by the phase function:

$$g = \int_{4\pi} P(\theta) \cos\theta \, d\Omega.$$

A plot of the asymmetry factor as a function of size parameter shows how forward scattering increases with particle size. In addition you can see that it increases with decreasing n_r , because the constructive interference of the diffraction ($l=0$) and refracted light ($l=2$) is enhanced (Fig. 6). Let's end this section with a Mie Scattering sunset (Fig. 7). Now what about this image is Mie?

Table 1: Indices of Refraction

Material	n_r (0.59 μm)
Air	1.0003
Methane	1.29
Water	1.33
Human Lenz	1.40
Crown glass	1.52
Diamond	2.42



Fig. 9.— Mie scattering in a sunset.