The Radiative Transfer Equation

\[ \frac{dI_\nu}{d\tau} = -I_\nu + S_\nu. \]
Effective temperature of a planet

• Assume that a planet radiates like a black body at a constant temperature, $T_{\text{eff}}$, where the solar energy absorbed by the planet equals that emitted.

• This temperature is called the planet’s effective temperature, $T_{\text{eff}}$

• How would you derive it?
Solid Angle: Ratio of an area on a sphere divided by the radius squared

\[ \Omega = \int \int_S \frac{\hat{r} \cdot d\hat{\mathbf{a}}}{r^2} \]

Unitless unit: steradian

This is 1 steradian
Solid Angle: Definition

Largest solid angle:

\[
\int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \sin(\theta) d\theta d\phi = 4\pi
\]

\sim 12.6
Solid angles on Earth

- 10 sr
- 1 sr
- 0.01 sr
- 0.1 sr
- 0.001 sr
Groups of photons

- Intensity

\[ I_{\lambda}(P, \hat{n}) = \frac{dE_\nu}{dA \, d\Omega \, d\lambda \, dt}, \]

The point \( P \) is surrounded by a small element of area \( d\pi_s \) perpendicular to the direction of the unit vector \( \mathbf{s} \). From each point on \( d\pi_s \) a cone of solid angle \( d\omega \) is drawn about the \( \mathbf{s} \) vector. The bundle of rays, originating on \( d\pi_s \) and contained within \( d\omega \), transports in time \( dt \) and in the wavelength range \( \lambda \) to \( \lambda + d\lambda \), the energy

\[ dE_\lambda = I_\nu(P, \hat{n}) \, dA \, d\Omega \, d\lambda \, dt \]

Here, \( d\Omega = d\omega = \text{solid angle} \)
What if you want a measure of the insolation that heats the surface of a planet?
Groups of photons

- Intensity

\[ I_\lambda(P, \hat{n}) = \frac{dE_\nu}{dA
d\Omega
d\lambda
dt}, \]

What if you wanted to know the flux?
That is the total number of photons absorbed per unit surface area per second per unit wavelength?

What is the Sun’s intensity?
What is its flux?
Groups of photons

- Intensity

- Flux

\[ dE_\lambda = I_\nu(P, \hat{n}) \, dA \, d\Omega \, d\lambda \, dt \]

\[ I_\lambda(P, \hat{n}) = \frac{dE_\nu}{dA \, d\Omega \, d\lambda \, dt} \]

\[ F_\nu(P, \hat{r}) = \int I(P, \hat{n}) \,(\hat{r} \cdot \hat{n}) \, d\Omega \]

What is the Sun’s intensity?
What is its flux?

\[ (\hat{r} \cdot \hat{n}) = \cos(\theta) \]
More definitions

- Intensity:

- Mean Intensity:

- Photon moves \( ds \) in \( dt = ds/c \)

- Since:

- Write energy transport as:

- \( E \) density/steradian:

- Energy density is then
Absorption, scattering & emission

• Extinction: an interaction between light & matter that decreases the intensity

\[ dI_\nu = -(\kappa_a + \kappa_s)_\nu \rho I_\nu ds \]

\[ dI_\nu = -(\sigma_a + \sigma_s)_\nu n I_\nu ds \]

• Emission adds to intensity:

Can be due to light scattered back into beam!

\[ dI_\nu = +j_\nu \rho ds \]

• Equation of Transfer:

\[ \frac{1}{\rho} \frac{dI_\nu}{ds} = -(\kappa_a + \kappa_s)I_\nu + j_\nu \]
The Source Function

- **Equation of Transfer:**

\[
\frac{1}{\rho} \frac{dI_\nu}{ds} = -(\kappa_a + \kappa_s)I_\nu + j_\nu
\]

- **Definition of Source Function:**

\[
S_\nu = \frac{j_\nu}{(\kappa_a + \kappa_s)}
\]

- **Optical Depth:**

\[
\tau = \int \kappa_\nu \rho ds.
\]

- **Simplified Equation:**

\[
\frac{dI_\nu}{d\tau} = -I_\nu + S_\nu.
\]

\(\rho ds\) is the column abundance
## Integrated Column Abundances

<table>
<thead>
<tr>
<th>Planet</th>
<th>$P_S$ (bar)</th>
<th>$T_S$ (K)</th>
<th>$g$ (m/s$^2$)</th>
<th>$U$ (km am)</th>
<th>$H$ (km)</th>
<th>$\Gamma$ (K/km)</th>
<th>Composition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth</td>
<td>1.01</td>
<td>288</td>
<td>9.8</td>
<td>8.1</td>
<td>8.5</td>
<td>9.8</td>
<td>78% $N_2$, 21% $O_2$</td>
</tr>
<tr>
<td>Venus</td>
<td>92</td>
<td>755</td>
<td>8.9</td>
<td>540</td>
<td>15.9</td>
<td>10.5</td>
<td>96.5% $CO_2$, 3.5% $N_2$</td>
</tr>
<tr>
<td>Mars</td>
<td>0.008</td>
<td>210</td>
<td>3.7</td>
<td>0.1</td>
<td>11.1</td>
<td>4.5</td>
<td>95.3% $CO_2$, 2.7% $N_2$</td>
</tr>
<tr>
<td>Titan</td>
<td>1.45</td>
<td>94.5</td>
<td>1.35</td>
<td>87</td>
<td>20</td>
<td>1.3</td>
<td>95% $N_2$, 3% $CH_4$</td>
</tr>
</tbody>
</table>
General Solution

• Equation of Transfer:

\[ \frac{dI_\nu}{d\tau} = -I_\nu + S_\nu. \]

How would you solve this equation, i.e. get the intensity terms on one side of the equation?
General Solution

- Equation of Transfer:

\[
\frac{dI_v}{d\tau} = -I_v + S_v
\]

- Multiply both sides by \(e^{\tau}\)

- General Solution:

\[
I_v(\tau_v) = I_v(0) \ e^{-\tau_v} + \int_0^{\tau_v} S_v e^{-(\tau_v-\tau') \ d\tau'}
\]
What does this mean?

\[ I_\nu(\tau_\nu) = I_\nu(0) \, e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu e^{-(\tau_\nu - \tau'_\nu)} \, d\tau' \]
Found Simplest Solution

\[ I_\nu(\tau_\nu) = I_\nu(0) \, e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu e^{-\left(\tau_\nu - \tau_\nu'\right)} \, d\tau' \]

No Source function
No source

- Equation of Transfer:

\[ I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu e^{-(\tau_\nu - \tau'_\nu)} d\tau' \]

- If S=0:

\[ I_\nu(s) = I_\nu(0) e^{-\tau} \]
Mean free path of a photon

- $e^{-\tau(s)} \propto$ probability that the photon makes it to a distance $s$.
- Normalize to sum of all the probabilities that it goes any distance to get the probability that the particle is absorbed between $s$ & $s+ds$:

$$P(s) = \frac{e^{-\tau_s}}{\int_0^\infty e^{-\tau_{s'}} ds'}$$

The expected (or mean) value of $s$ is then

$$\langle s \rangle = \int_0^\infty s P(s) ds = \frac{1}{\rho \kappa_{\nu}}$$

Mean free path: average distance before being absorbed or scattered

**Bouguer’s law is then:** $I = I_0 e^{-s/\langle s \rangle}$
Beer’s law or Lambert’s Law, but really Bouguer’s law...

\[ I_\nu (\tau_\nu) = I_\nu (0) \ e^{-\tau_\nu}. \]
Beer’s law or Lambert’s Law, but really Bouguer’s law...

\[ I_\nu(\tau_\nu) = I_\nu(0) \ e^{-\tau_\nu} \ . \]

\[ \tau = \int \kappa_\nu \rho \, ds \]
Beer’s law or Lambert’s Law, but really Bouguer’s law...

\[ I_\nu(\tau_\nu) = I_\nu(0) \ e^{-\tau_\nu}. \]

Transmission: \( T = e^{-\tau} \)

Absorption: \( A = 1 - e^{-\tau} \)
Apply Beer’s law to an atmosphere

\[ I_\nu(\tau_\nu) = I_\nu(0) \ e^{-\tau_\nu} \]

\[ \tau = \int \kappa_\nu \rho ds \]

\[ I_{1\lambda} = I_\lambda e^{-\tau} \]

<table>
<thead>
<tr>
<th>\rho_1</th>
<th>\tau_1</th>
<th>I_{1\lambda}</th>
</tr>
</thead>
<tbody>
<tr>
<td>\rho_2</td>
<td>\tau_2</td>
<td></td>
</tr>
<tr>
<td>\rho_3</td>
<td>\tau_3</td>
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<tr>
<td>\rho_7</td>
<td>\tau_7</td>
<td></td>
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Use Beer’s law to calculate the transmission through Earth’s atmosphere.