They already knew:
Total power/surface area

\[ P(T) = \sigma T^4 \]
\[ \sigma = 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4} \]

Also:

\[ \lambda_{MAX} = \frac{b}{T} \]
\[ b = 2.9 \times 10^{-3} \text{ m K} \]

But what is \( B_{\nu}(T) \)?

Question: what is the nature of radiation emitted by an object in equilibrium
A blackbody

- Body in *thermodynamic equilibrium*: i.e. in chemical, mechanical, radiative equilibrium
- Temperature is constant
- Intensity depends only on temperature
- Temperature depends on intensity
- If $I_\nu(A) > I_\nu(B) \rightarrow$ B will heat & A will cool. But that’s a contradiction: they’re same temperature

Assume 2 blackbodies of same temperature — they emit the same intensity
Kirchhoff's law

- Source Function, $S = \text{Intensity, } B$
- If $S > B$ then $I > B$, but $T$ is the same everywhere

\[ \frac{j_\nu}{\kappa_\nu} = I_\nu \]
Derive the intensity of a black body

Strategy:

- Derive the density of states \( D_\nu \)
- Derive energy of each state \( E_\nu \)
- Derive energy density \( \mu_\nu \)
- Derive intensity \( I_\nu \)

\[ I_\nu = J_\nu = \frac{c}{4\pi} \mu. \]
Energy states

- Assume cavity of dimensions $L \times L \times L$
- E field must be 0 at boundary (otherwise energy dissipates out of cavity)
- Consider waves as a superposition of those in the $x$, $y$ and $z$ directions, $\hat{n}_x$, s.t. $k = \left(\frac{2\pi}{\lambda}\right)n$
Count the allowable states

- Density of States: assume a BB of dimensions $L \times L \times L$

- EM wave equation:

$$\frac{\delta^2 E}{\delta x^2} + \frac{\delta^2 E}{\delta y^2} + \frac{\delta^2 E}{\delta z^2} = \frac{1}{c^2} \frac{\delta^2 E}{\delta t^2}$$

- Solution ($E=0$ at walls):

$$E = E_0 \sin \frac{n_1 \pi x}{L} \sin \frac{n_2 \pi y}{L} \sin \frac{n_3 \pi z}{L} \frac{2 \pi c t}{\lambda}$$

- Where allowed frequencies are:

$$\left[\frac{n_1 \pi}{L}\right]^2 + \left[\frac{n_2 \pi}{L}\right]^2 + \left[\frac{n_3 \pi}{L}\right]^2 = \left[\frac{2 \pi}{\lambda}\right]^2$$

- Lattice points in a sphere of radius $r$:

$$n_1^2 + n_2^2 + n_3^2 = \frac{4L^2}{\lambda^2}$$

$$r^2 = n_1^2 + n_2^2 + n_3^2 = \frac{4L^2 \nu^2}{c^2}$$

Then: $r = 2L \nu / c$
Density of States (1-D)

Number of allowed states within \( \nu \) and \( d\nu \):

From:

\[
\nu = \frac{nc}{2L}
\]

Number of states is:

\[
N(\nu) = \left( \frac{2L}{c} \right)(\nu + d\nu) - \left( \frac{2L}{c} \right)\nu = \left( \frac{2L}{c} \right) d\nu
\]

Thus:

\[N(r)dr = N(\nu)d\nu\]
Allowable states

- For 3-D you have a 3-D lattice such that the points are defined on a shell with a width of \( dv \)
- \( N(\nu)dv = N(r)dr = 4\pi r^2 \, dr \)
- But only consider positive values:
  - \( N(\nu)dv = 1/8 \cdot 4\pi r^2 \, dr = 1/2 \pi r^2 \, dr \)
- Substitute for \( r \):
  \[ N(\nu)dv = \frac{\pi}{2} \left( \frac{2L}{c} \right)^3 \nu^2 dv \]
- Multiply by 2 for 2 polarization states
  \[ N(\nu)dv = \frac{8\pi V}{c^3} \nu^2 dv \]
- Each mode has an energy of \( kT \) (equipartition of energy for 2 degrees of freedom). The energy density is thus:
  \[ \mu_\nu = \frac{8\pi k_B T}{c^3} \nu^2 \]
  \[ B_\nu = I_\nu = \frac{c}{4\pi} \frac{8\pi \nu^2}{c^3} k_B T \]
Rayleigh – Jeans Law

\[ B_\nu = I_\nu = \frac{c}{4\pi} \frac{8\pi \nu^2}{c^3} k_B T \]

Any problems here?
"Ultraviolet Catastrophy"

- Rayleigh-Jeans law doesn’t make sense!
- Doesn’t agree with the oven measurements!
Towards an explanation

- Average energy according to Boltzmann Distribution:
  \[ P(E) = \frac{e^{E/k_BT}}{k_BT} \]
  \[ \bar{E} = \frac{\int_0^\infty EP(E)dE}{\int_0^\infty P(E)dE} = k_BT \]

- If the states could have discrete energies, e.g. 0, \(\Delta E\), 2\(\Delta E\)... Then
  - If \(\Delta E \ll kT\): \(\bar{E} \sim kT\)
  - If \(\Delta E \gg kT\): \(\bar{E} < kT\)
Planck’s desperate idea

- Let the interval $\Delta E$ of allowed states depend on $v$

- Let’s try $\Delta E = hv$ and derive $h$ from the intensity (i.e. less energy from the high $v$ light)

- Fit the data: get $h = 6.63 \times 10^{-34}$ J sec

- Actual value of “Planck’s constant”: $6.626068 \times 10^{-34}$ J s.

- Oven experiment indicates that light comes only in quantized packets!
The probability of finding a state of energy \( E_j \), \( P(E_j) \), is proportional to \( e(-E_j/kT) \).

Where does this general statement come from?

It is proportional to the number of states having this energy & number of states accessible to larger system having energy \( E-E_j \):

\[
P_j \propto N(E^0 - E_j)
\]

Work with natural log because #states depends on \( E \). Expand log of number

\[
E_j \ll E^0 \quad \rightarrow \quad \ln N(E^0 - E_j) = \ln N(E^0) + \left[ \frac{d\ln N'}{dE'} \right]_0 (-E_j)
\]

From thermodynamics: \( \beta = 1/kT \)

\[
\ln N(E^0 - E_j) = (\ln N(E^0)) - \beta E_j.
\]

\[
P(E_j) = N(E^0 - E_j) = C e^{-\beta E_j}
\]

Let \( g_j \) be the # of states with energy \( E_j \) i.e. the degeneracy

\[
P(E_j) = \frac{g_j e^{-\beta E_j}}{\sum_0^\infty g_i e^{-\beta E_i}}
\]
Types of particles

- Maxwell–Boltzmann statistics
  - Distinguishable
  - Each particle has an energy
  - Obey classical mechanics

- Fermions
  - Energy, location & state not specified but by probability
  - \(\frac{1}{2}\) integral spin
  - Obeys Fermi’s law: only one per state
    - Protons, neutrons, electrons, quarks..

- Bosons
  - Also quantum mechanical
  - Integer spin
  - More than one can exist in a state
  - Force carriers (photons, gluons, gravitons & W & Z bosons)
    - For photons – there are an unspecified # of particles
Mean number of photons in a state

- Mean number of photons in a state: Sum states weighted by probability that it occurs

\[
\bar{n}_s = \frac{\sum n_1, n_2 \ldots n_s n_s e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \ldots)}}{\sum n_1, n_2 \ldots n_s e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \ldots)}}
\]

- OR

\[
\bar{n}_s = \frac{\sum n_s e^{-\beta n_s \epsilon_s} \sum_{n_1, n_2 \ldots} e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \ldots)}}{\sum_{n_s} e^{-\beta n_s \epsilon_s} \sum_{n_1, n_2 \ldots} e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \ldots)}}
\]

- A photon can have all possible states regardless of occupation

“-s” means that state “s” is not in sum
Mean number of photons in a state

- A photon can have all possible states regardless of occupation

\[ \bar{n}_s = \frac{\sum n_{1,n_2...n_s} n_s e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + ...)}}{\sum n_{1,n_2...n_s} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + ...)}} \]

- OR

\[ \bar{n}_s = \frac{\sum n_s e^{-\beta n_s \epsilon_s} \sum_{n_{1,n_2...n_s}} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + ...)}}{\sum n_s e^{-\beta n_s \epsilon_s} \sum_{n_{1,n_2...n_s}} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + ...)}} \]

\[ \bar{n}_s = \frac{\sum n_s n_s e^{-\beta n_s \epsilon_s}}{\sum n_s e^{-\beta n_s \epsilon_s}} \]
Mean number of Particles in a State

Now just a little math

\[ \bar{n}_s = \frac{\sum n_s e^{-\beta n_s \epsilon_s}}{\sum n_s e^{-\beta n_s \epsilon_s}} \]

\[ \bar{n}_s = \frac{-(1/\beta) \frac{d}{d\epsilon_s} \sum e^{-\beta n_s \epsilon_s}}{\sum e^{-\beta n_s \epsilon_s}} = -\frac{1}{\beta} \frac{d}{d\epsilon_s} \ln(\sum e^{-\beta n_s \epsilon_s}) \]

\[ \bar{n}_s = \frac{1}{\beta} \frac{d}{d\epsilon_s} \ln(1 - e^{-\beta \epsilon_s}) = \frac{e^{-\beta \epsilon_s}}{1 - e^{-\beta \epsilon_s}} \]

\[ \bar{n}_s = \frac{1}{e^{\beta \epsilon_s} - 1} \]
Calculate the average energy of radiation of frequency $\nu$:
Designate each state, $n$, be that of $n$ photons each with $E = h\nu$
The $n$th state is $E_n = nh\nu$
The mean energy is the sum weighted by probability that energy occurs

$$\bar{E}(\nu) = \frac{\sum_{n=0}^{\infty} nh\nu e^{-\beta nh\nu}}{\sum_{n=0}^{\infty} e^{-\beta nh\nu}}$$

That is just*:

$$\bar{E}(\nu) = h\nu\bar{n} = \frac{h\nu}{e^{\beta h\nu} - 1}$$

The energy density is then

$$\mu = \frac{8\pi \nu^2 h\nu}{c^3(e^{h\nu/k_BT} - 1)}$$

The blackbody intensity:

$$B_\nu = I_\nu = \frac{c}{4\pi} \mu = \frac{2h\nu^3}{c^2(e^{h\nu/k_BT} - 1)}$$

* Not KT
**Blackbody Intensity**

- **Average energy:**
  \[
  \bar{E}(\nu) = h\nu\tilde{n} = \frac{h\nu}{e^{\beta h\nu} - 1}
  \]

- **Derive \( h \) from measurements:**
  \[h = 6.63 \times 10^{-34} \text{ Jsec.}\]

- **Derived intensity:**
  \[
  B_\nu = I_\nu = \frac{c}{4\pi}\mu = \frac{2h\nu^3}{c^2(e^{h\nu/k_BT} - 1)}
  \]

- **Compared to classical:**
  \[
  I_\nu = \frac{c}{4\pi} \frac{8\pi\nu^2}{c^3} k_BT.
  \]
Results presented at the German Physical Society on December 14, 1900

A good reference to thermodynamics and black-body radiation is F. Rief, Fundamentals of Statistical and Thermal Physics.

Nobel Prize in 1918