## 1. Runaway Greenhouse

*"Making the simple complicated is commonplace; making the complicated simple, awesomely simple, that's creativity." - Charlie Mingus (Jazz Musician)*

Consider Venus' atmosphere, which, because its thick sulfuric acid ($H_2SO_4$) clouds, absorbs less solar insolation than Earth. Venus' bond albedo is 0.9 and Earth's is 0.3, and the equivalent blackbody temperature of these planets is 184 K and 255 K respectively. Despite, the fact that Venus has a lower equivalent temperature, it's surface temperature is 870 F (i.e. 465 C and 735 K). That is, it is ridiculously higher than Earth's average surface temperature of 60 F (i.e. 15 C and 288 K). This comparison brings up two questions: 1) How can we best understand the underlying physics of the greenhouse effect that causes this great discrepancy in the surface temperatures of Earth and Venus; and 2) how did Venus evolve to it's present state? It seems that to understand the nature of planetary atmospheres we would like to at least understand why Earth's very close twin, Venus, which has a similar density, size, distance from it's host star (the Sun) and presumably bulk composition, ended up so different from Earth.

Towards these ends we will use a very simple model of an atmosphere to study thermal profiles of tropospheres. Let's assume that the atmosphere that is transparent at visible wavelengths and yet opaque at the longer wavelengths emitted by planets. Essentially, we assume that the atmosphere is heated only from below by a warm surface. This is a fair assumption for Earth. We further assume that the atmosphere is plane-parallel, optically grey, does not scatter, and thermally emits according to LTE. This model does not reproduce any atmosphere very well, yet it provides a tool for understanding the runaway greenhouse effect. We'll read Andy Ingersoll's seminal paper that illuminates the runaway greenhouse effect on Venus, before people even knew the composition of Venus' clouds.

### 1.1. A surface-heated 2-stream atmosphere

This section will follow the arguments in Chamberlain and Hunten's book, *Theory of Planetary Atmospheres.*

Initially we assume that the radiative solution is convectively stable. As such, it's profile depends on the ground temperature and the opacity of the atmosphere as manifested by its composition.

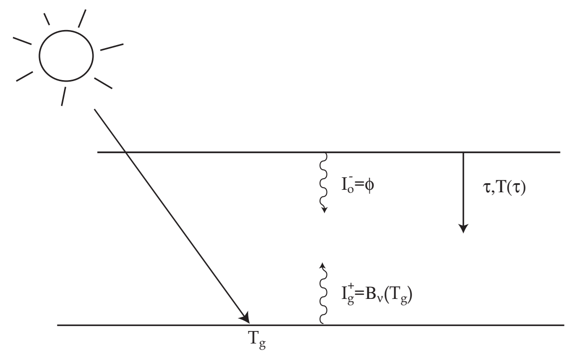


Fig. 1.— **An atmosphere transparent to visible light and opaque to infrared radiation.**

Note this model treats only the infrared emission of the planet. We assume a grey atmosphere, i.e. that having only one absorption coefficient. This mean absorption coefficient for the atmosphere can be derived by averaging (in a weighted sense) the absorption of the gases over the spectral region defined by the blackbody emission of the planet. We will also assume a *two stream* approximation, where the radiation field is broken into two streams: an upward one and a downward one.

The equation of radiative transfer for a plane parallel atmosphere in local thermodynamic equilibrium is:

$$\frac{dI_\nu}{d\tau} = \frac{\mu dI_\nu}{\kappa\rho ds} = -I_\nu + B_\nu(T), \tag{1}$$

where, in this case, $\tau$ depends only on distance $s$, and $\mu$=cos($\theta$) is the cosine to the vertical angle. The source function $S_\nu(T)$ is the planck function $B_\nu(T)$.

Recall that the mean intensity is defined is:

$$J_\nu = \frac{1}{4\pi}\int I_\nu d\Omega$$

. Also note that for a two stream model, the intensity is independent of the azimuthal angle. The net flux across an area parallel to the surface is then:

$$F_\nu = \int_0^{2\pi}\int_{-1}^{1} I_\nu(\mu)\mu\ d\mu d\phi = 2\pi\int_{-1}^{1} I_\nu(\mu)\mu\ d\mu. \quad (2)$$

Now we will work with coordinates of altitude, $z$, which increase in the opposite direction from the optical depth, which is defined relative to the top of the atmosphere where it has its lowest value. Then $dz =$ -$ds$, where $ds$ is used in the original formulation of the radiative transfer equation and has a value which increases with $\tau$. Thus:

$$d\tau = -\kappa\rho dz,$$

and the RT equation, thereby incorporating the sign change, is from Eq. 1:

$$\frac{\mu dI_\nu}{d\tau} = I_\nu - B_\nu(T). \quad (3)$$

Lets derive a few relationships between $F_v$, $J_v$ and $B_v$. First integrate the RT equation (Eq. 3) over a sphere:

$$\frac{d}{d\tau_\nu}(F_\nu) = 4\pi(J_\nu - B_\nu) \quad (4)$$

In the *two stream approximation*, we define the upward intensity be:

$$I^+(\tau): \qquad 0 < \mu < +1 \quad (5)$$

and the downward intensity be:

$$I^-(\tau): \qquad -1 < \mu < 0. \quad (6)$$

The mean intensity at a depth of $\tau$ is

$$J_\nu = \frac{1}{2}\int_{-1}^{1} I_v(\mu)d\mu = \frac{1}{2}(I^+ + I^-) \quad (7)$$

and the net flux is:

$$F_\nu = 2\pi[\int_{-1}^{0} I^-\mu d\mu + \int_0^1 I^+\mu d\mu] = \pi(I^+ - I^-) \quad (8)$$

Now we obtain a second relation between the mean intensity and the flux by multiplying the RT equation (Eq. 3) by $\mu$ and integrating over the sphere. Effectively, here we're just isolating the directionally dependent terms.

$$\int_0^{2\pi}\int_{-1}^{1}\mu^2\frac{dI_\nu}{d\tau}\ d\mu d\phi = \int_0^{2\pi}\int_{-1}^{1} I_\nu\mu\ d\mu d\phi$$

$$-\int_0^{2\pi}\int_{-1}^{1} B_\nu(T)\mu\ d\mu d\phi \quad (9)$$

The last term is zero because the Planck source emits isotropically. The first term on the right is the net flux. Since this is a two stream calculation, the intensity is independent of the azimuthal angle $\phi$. Thus equation 9 becomes:

$$2\pi\int_{-1}^{1}\mu^2\frac{dI_\nu}{d\tau}\ d\mu = F_\nu.$$

The left term is:

$$2\pi\ \frac{d}{d\tau_\nu}\ [\int_{-1}^{0} I^-\mu^2 d\mu + \int_0^1 I^+\mu^2 d\mu]$$

or

$$2\pi\ \frac{d}{d\tau_\nu}\ [\ I^-\frac{\mu^3}{3}|_{-1}^{0} + I^+\frac{\mu^3}{3}|_0^1\ ]$$

or, noting equation 7:

$$\frac{2\pi}{3}\ \frac{d}{d\tau_\nu}\ (I^- + I^+) = \frac{4\pi}{3}\ \frac{dJ_\nu}{d\tau_\nu}$$

Equation 9 becomes:

$$\frac{4\pi}{3}\frac{dJ_\nu}{d\tau_\nu} = F_\nu,$$

and substituting for $J_\nu$ (eq. 4) yeilds,

$$-\frac{d^2F_\nu}{d\tau_\nu^2} - 3F_\nu = -4\pi\frac{dB_\nu}{d\tau_\nu} \quad (10)$$

### 1.2. Grey Atmosphere & Radiative Equilibrium

Now that we have simplified the radiative transfer equation (Eq. 10) with the 2-stream and an isotropic source function, we invoke radiative equilibrium to model for the temperature profile.

First, integrate the radiative transfer equation,

$$\frac{\mu dI_\nu}{\kappa_\nu\rho dz} = I_\nu - B_\nu,$$

over all frequencies:

$$\frac{\mu}{\rho}\frac{d}{dz}\left(\int_0^\infty \frac{I_\nu}{\kappa_\nu}dv\right) = I - B \tag{11}$$

where,

$$I = \int_0^\infty I_\nu d\nu, \qquad B = \int_0^\infty B_\nu d\nu. \tag{12}$$

We will assume that at every level in the atmosphere the flux into the layer equals that coming out. This is the condition of radiative equilibrium. That is, assume that the net flux is conserved:

$$F = \int_0^\infty F_\nu \; d\nu = \text{constant}.$$

Then, from Eq. 10:

$$-\frac{1}{\rho}\frac{d}{dz}\left(\int_0^\infty \frac{B_\nu}{\kappa_\nu}d\nu\right) = \frac{3}{4\pi}F. \tag{13}$$

If the absorption coefficient is constant with wavelength, the $\kappa$ comes out of the integral and:

$$\frac{dB}{d\tau} = \frac{3}{4\pi}F. \tag{14}$$

For a non-grey atmosphere we might be able to approximate the absorption coefficient as a weighted average. Such an average suggests itself by a comparison of eqs. 12 and 13:

$$\frac{1}{<\kappa>} = \frac{1}{B}\int_o^\infty \frac{B_v}{\kappa_v}dv. \tag{15}$$

This approximate absorption coefficient, called the *Rossland mean*, works fairly well if the vertical temperature structure does not shift the frequency distribution of radiation.

Alternatively we can use the *Chandrasekhar mean*

$$<\kappa> = \frac{1}{F}\int_o^\infty \kappa_v F_v dv \tag{16}$$

however the flux would need to be solved iteratively.

### 1.3. Boundary Conditions

Assume an atmosphere that radiates outward above a ground that is a black body of temperature $T_g$. In addition, the top of the atmosphere is met with a cold sky, with zero downward intensity. These are our boundary conditions.

First note that we can get rid of $J$ (not an easy variable to determine physically) by expressing this term by (note Eqs. 7-8):

$$J = I^- + \frac{1}{2\pi}F = I^+ - \frac{1}{2\pi}F. \tag{17}$$

Now, the RT equation (Eq. 4) is expressed as

$$\frac{dF}{d\tau} = 4\pi(I^+ - B) - 2F$$

$$= 4\pi(I^- - B) + 2F = 0.$$

These equations give us expressions for the upperward ($I^+$) and downward ($I^-$) intensities:

$$I^+ = B(T) + \frac{1}{2\pi}F \tag{18}$$

$$I^- = B(T) - \frac{1}{2\pi}F \tag{19}$$

Now, evaluate the *boundary conditions*, considering Eq. 8. At the ground:

$$I_g^+ = B(T_g) = B(T_1) + \frac{1}{2\pi}F, \tag{20}$$

where $T_1$ is the temperature of the atmosphere above the ground. At the top of the atmosphere, from Eq. 19,

$$I_0^- = 0 = B(T_0) - \frac{1}{2\pi}F,$$

or,

$$\frac{1}{2\pi}F = B(T_0). \tag{21}$$

We find then that an LTE solution for a grey atmosphere results in a discontinuity at the ground with an atmospheric temperature of $T_1 < T_g$. The upward flux from the top of the atmosphere is (from Eqs. 18 & 20):

$$\pi I_0^+ = \pi B(T_0) + \frac{1}{2}F = 2\pi B(T_0)$$

That is, the upward intensity at the top of the atmosphere is

$$I_0^+ = 2B(T_0) \tag{22}$$

And since $F$ is a constant, from Eq. 14,

$$B(\tau) - B(T_0) = \frac{3}{4\pi}F\tau$$

Therefore, from the boundary condition (eq. 21):

$$B(\tau) = B(T_0)(1 + \frac{3}{2}\tau). \quad (23)$$

Thus the air at the top of the atmosphere radiates with a value of $T_0$. You can see from the equation above (Eq. 22–23) that the upward flux ($I_0^+$=$2B(T_0)$) is characteristic of the thermal emission at $\tau = \frac{2}{3}$.

### 1.4. Temperature Structure

From the Stefan-Boltzmann law and Eq. 23:

$$T^4(\tau) = T_0^4(1 + \frac{3}{2}\tau). \quad (24)$$

where, $T_0$ is the temperature of the upper boundary, and the integrated black-body intensity is

$$B(\tau) = \frac{\sigma}{\pi}T^4(\tau)$$

The radiant flux from a planet can be expressed as a mean emission temperature called the *equivalent temperature*, $T_e$,the temperature that (by definition) corresponds to that of the black body that radiates with the same flux, $F$, as the planet. From Eq. 21 and that above we find:

$$T_e = (\frac{F}{\sigma})^{1/4} = (\frac{2\pi B(T_0)}{\sigma})^{1/4} = 2^{1/4}T_0.$$

i.e.

$$T_e^4 = 2T_0^4$$

If we assume that we have a (rotating) planet heating uniformly across its disk, the emitted flux is equal to the absorbed incident solar flux, in a steady state situation.

$$4\pi R^2\sigma T_e^4 = (1 - A)\pi R^2(F_\odot) \quad (25)$$

That is:

$$T_e = [\frac{(1 - A)}{4\sigma}F_\odot]^{\frac{1}{4}}$$

where $A$ is the effective planetary albedo, and $F_\odot$ is the incident solar flux. Note that if we assume a reasonable value for the albedo of Earth, $A$=0.29, we derive an effective temperature of $T_e$=255 K and a "skin temperature" of $T_o$=215 K which is very close to the mid latitude tropopause temperature. Now, If we know the optical thickness verses height, we can figure out the temperature profile of the atmosphere - that is the lapse rate for radiative equilibrium.

The discontinuity between the air and surface temperatures can be derived from the boundary condition (eq. 20), again integrated over the frequency

$$T_g^4 = T_1^4(\tau_g) + \frac{1}{2}T_e^4,$$

or

$$B(T_g) = B(T_1) + B(T_0) \quad (26)$$

where the later term $B(T_0) = \frac{F}{2}$ is the difference between the intensity at the surface $B(T_g)$ and intensity of the air right above the surface $B(T_1)$. The surface temperature, $T_g$, being larger than $T_1$, involves a jump in temperature compared to the thermal profile represent that represent the atmosphere (Eqs. 23 and 24).

The intensity of the air above the surface $B(T_1)$ is described by the expression

$$B(T_1) = B(T_0)(1 + \frac{3}{2}\tau_g), \quad (27)$$

where $\tau_g$ is the total optical depth of the atmosphere. Thus, the temperature of the air directly above the surface is:

$$T_1^4 = T_0^4\ (1 + \frac{3}{2}\tau_g).$$

The blackbody intensity of the surface itself exceeds that of the air above (Eq. 27) by an additive term of $B(T_0)$, as shown in equation 26:

$$B(T_g) = B(T_0)(1 + \frac{3}{2}\tau_g) + B(T_0)$$

i.e.,

$$B(T_g) = B(T_0)(2 + \frac{3}{2}\tau_g).$$

The temperature of the surface is therefore:

$$T_g^4 = T_0^4(2 + \frac{3}{2}\tau_g), \quad (28)$$

and, since $T_e^4/2 = T_0^4$, then

$$T_g^4 = T_e^4(1 + \frac{3}{4}\tau_g), \quad (29)$$

where $\tau_g$ is the optical thickness at the ground. Note that for this example the flux is constant throughout the atmosphere, and equal to that emitted at the top of the atmosphere. Note also that the ground temperature can be high for an atmosphere with a large optical depth $\tau$. This is a simple formalism of the *greenhouse effect.*

## 2. Runaway Greenhouse Effect

In 1969, Andy Ingersol published a seminal paper on the "runaway greenhouse" to explain how Venus' atmosphere evolved drastically different from that of Earth. He started with the *Eddington Approximation* temperature profile derived above and included a few assumptions concerning Venus's early condition (Ingersol, A. P., *J. of Atmospheric Sc.* **26** p. 1191, 1969). The stratosphere is assumed to heated from below by solar radiation absorbed by the surface. The infrared transparency is assumed to be governed by a single gas, at equilibrium with its liquid at the surface. The resulting optical depth is approximated with a absorption coefficient, $\kappa$, that is independent of wavelength:

$$\tau = \int_z^\infty \kappa \rho_v dz \tag{30}$$

where $\rho_v$ is the mass density of the condensable gas. The radiative equilibrium solution is (note eq. 23):

$$B(\tau) = \frac{F}{2\pi}(1 + \frac{3}{2}\tau) = \frac{\sigma}{\pi}T^4(\tau), \tag{31}$$

where,

$$B(T_0) = \frac{F}{2\pi}. \tag{32}$$

At the ground (Eq. 28):

$$B_g(\tau_g) = \frac{F}{2\pi}(2 + \frac{3}{2}\tau_g),$$

which results from the surface discontinuity, that is:

$$B_g(\tau_g) > B(\tau_g).$$

For a solar constant, S = $F_\odot$ = 2 cal cm$^{-2}$ min$^{-1}$, and an albedo of 0.4, the global average flux of sunlight absorbed by the Earth is (1-A)S/4 = 0.3 cal cm$^{-2}$ min$^{-1}$. This must equal the outward flux from the planet, $F$, if the planet is in thermal equilibrium. If the optical depth is of order 1, then the surface temperature is 280-300 K, good for Earth.

Let's now place this stratosphere (or the radiative part of the atmosphere) above a troposphere with ocean reservoirs of the main constituent of the atmosphere (like water on Earth), such that the dynamics of moist convection insures that the troposphere be close to saturation at all levels.

From hydrostatic equilibrium,

$$\rho \, dz = -\frac{dP}{g}. \tag{33}$$

Now assume that the volume mixing ratio $q_v$ of the vapor and its absorption coefficient $\kappa$ are constant. Then the mass density of the vapor can be written in terms of the atmospheric mass density, $\rho$:

$$\rho_v = q_v(\frac{m_v}{m})\rho, \tag{34}$$

where $m_v$ is the mean molecular weight of the vapor and $m$ is the mean molecular weight of the atmosphere. From Eq. 30, 33 and 34:

$$\tau = \kappa q_v \frac{m_v}{m} \int_z^\infty \frac{dP}{g}$$

Assuming $g$ is also constant and integrating:

$$\tau = \kappa q_v \frac{m_v}{m} \frac{P}{g}$$

It follows that:

$$\tau = \kappa P_v m_v / mg, \tag{35}$$

where $P_v$=$Pq_v$ is the partial pressure of the vapor. In the radiative-convective boundary, the partial pressure, $P_v$, equals the saturation pressure $P_s(T)$ multiplied by the relative humidity, $H$.

$$P_v(T) = HP_s(T), \tag{36}$$

and $\tau$ can be written as:

$$\tau = \kappa H P_s(T) m_v / mg. \tag{37}$$

The outgoing flux can be expressed in terms of the temperature at the radiative-convective boundary ($T_{rc}$), which defines the base of the radiative stratosphere. In essence this functions as $T_0$. Thus, combining Eqs. 31 and 37:

$$\frac{F}{2\pi} = B(\tau)/[(1 + \frac{3}{2}\tau),$$

or,

$$\frac{F}{2\sigma} = T^4/[(1 + \frac{3}{2}\tau).$$

Considering Eq. 37, and evaluating the equation at the radiative-convective boundary:

$$\frac{F}{2\sigma} = T_{rc}^4[1 + \frac{P_s(T_{rc})}{P_0}]^{-1}, \tag{38}$$

where $P_0$ is defined as,

$$P_0 = \frac{2mg}{3\kappa H m_v}. \tag{39}$$

Ingersoll notes that the right hand side of the equation 38 has a maximum value, because for small $T$ it is proportional to $T^4$, and thus increases with $T$. For large $T$, it is proportional to $T^4/P_s(T)$ and thus decreases with $T$. Solutions therefore exist only for values less the maximum value. This critical value depends on the atmospheric composition ($P_s$ and $\kappa$). That is: there is a maximum flux that the planet can emit. Equilibrium is therefore impossible when the incident solar flux exceeds the critical value planetary outward flux. In this case the incoming solar flux, exceeding the outgoing flux, would supply energy to the surface to heat it. If there were oceans, this energy would go into evaporating the oceans. As Andy Ingersoll puts it: "This is the runaway greenhouse effect".

As an illustration, Ingersoll considers, among other options, an absorption coefficient of $\kappa = 0.1$ cm$^2$ gm$^{-1}$, appropriate for the 8-20 $\mu$m window. In addition, he assumes a saturated atmosphere ($H$=100%), the mean molecular weight typical of Earth and molecular nitrogen ($m$=29 g), water vapor ($m_v$=18 g), and a gravity typical of Earth and Venus ($g$=10 m/s$^2$). From these assumptions, he finds that $P_0$= 8 mm Hg, $F_{max}$=0.63 cal cm$^{-2}$ min$^{-1}$, at $T$=260. This can be compared to the average incident sunlight on Earth and Venus of 0.50 and 0.95 cal cm$^{-2}$ min$^{-1}$. Because of the higher reflectivity of Venus, both of these atmospheres absorb similarly 0.3 cal cm$^{-2}$ min$^{-1}$. Thus presently, given the 0.78 and 0.3 albedos for Venus and Earth respectively, both planets are sub-critical.

But suppose that Venus' clouds were not as optically thick as they are now, and suppose that an ancient Venus had water on its surface, as does Earth. After all these planets formed close to one another, and have the same density and size. One might therefore expect similar initial compositions. Let's assume that Venus atmosphere is such that the incident flux from the Sun exceeds that emitted. Then Venus' oceans will evaporate, and soon dominate the composition of the atmosphere, assuming an atmosphere like Earth (with $5\times10^{18}$ kg) and an ocean like Earth (with $1.4\times10^{21}$ kg). Note that the atmospheric mass could become rather large.

Now one, perhaps subtle, point about the runaway greenhouse effect, is that the outgoing radiation is set only by the upper atmosphere. Even if Venus' atmosphere becomes a monster water one, it still will have have a critical outgoing flux. Here the critical flux (i.e. $F_2$ in figure 2) would differ from that of current Earth because the opacity, and $m$, would be

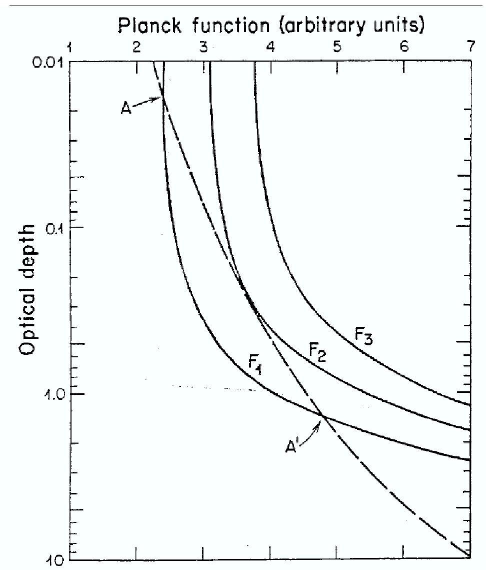


Fig. 2.— **A plot of the Planck function ($\sigma/\pi\ T^4$) verses optical depth. The solid lines are the relationship between the outgoing flux ($F$) and the temperature of that atmosphere in radiative equilibrium (Eq. 31). The dashed line is the relationship between the vapor pressure and the temperature, expressed as $\sigma/\pi\ T^4$ (Eq. 37). Note that if the outgoing flux is too high, e.g. $F_3$, there is no solution to equation 36. For too low a value, e.g. $F_1$, equation 36 has solutions only for an atmosphere that is supersaturated. And so here a convective troposphere is needed. The outgoing flux, $F_2$, represents the critical maximum outgoing flux capable of an atmosphere under the relationship (Eq. 38). From Ingersoll 1969.**

set by water. But there would still be one, and if the atmosphere could conceivably reach and surpass the critical point for a water atmosphere, then the runway greenhouse could prevail and create a water-based atmosphere. In essence runaway greenhouse atmospheres grows from below (Fig. 3).

Ingersoll evaluates the escape of water in Venus' atmosphere by considering an atmosphere consisting of water predominantly and calculating the mixing ratio of water as the water is cooled to stratospheric temperatures. He finds is that indeed for reasonable values of the temperature and for both water-rich and water-poor atmospheres, the runaway greenhouse can occur.

There is another interesting point in the paper. When water is the major constituent of the atmosphere, its partial pressure basically sets the atmosphere's pressure, and the water abundance remains high in the stratosphere. As a result, the water opacity (rather than that of $O_2$ and $O_3$) determines the penetration depth of solar UV radiation. In the process, water is dissociated, the hydrogen escapes, and the oxygen reacts with other molecules. Venus loses its water

This paper thereby shows that it is highly conceivable that Venus underwent a runaway greenhouse, and in the process lost its water. Notice the assumptions in the paper (e.g. the lack of clouds, the assumption of a grey atmosphere, the simplified atmospheric structure). Are they convincingly dealt with? Is it reasonable to calculate a more precise model? If so, how and why?

*C.A. Griffith class notes (Planetary Atmospheres 2015)*