NO. 60 THE REDUCTION OF MEASURES FOR POSITION ON A SINGLE LUNAR PHOTOGRAPH

by D. W. G. ARTHUR

July 5, 1965

ABSTRACT

The paper details the computational methods used at LPL to convert raw measures on lunar photographs into rectangular coordinates that are free of refraction.

1. Introduction

This paper describes the computational methods* used at LPL for the reduction of measurements for position on single lunar photographs. The principal purpose of these reductions is the determination of refraction-free photographic coordinates. The reductions also determine the selenographic positions of the observed points, or rather, the line-of-sight projections of these points on the mean lunar spherical datum.

To simplify the discussion, all description of the initial reductions of the photographic coordinates is omitted since this may vary considerably according to the measuring techniques. Instead, these initial reductions will be described in connection with each photograph.

It is assumed that a number of points are available whose selenodetic coordinates are known. At present there is some choice, since lists of such points have been published by G. Schrutka-Rechtenstamm (1958), by the Aeronautical Chart and Information Center of the U.S. Air Force (1965), and by the U.S. Army Map Service (1964). The Schrutka work is the most fundamental of these, and despite some defects in the materials and in their processing I have selected this list to control the LPL reductions.

The notation used in this paper is listed as follows:

(ξ, η, ζ) = standard selenodetic direction-cosines;

(E, F, G) = selenodetic rectangular coordinates in units of the moon's mean radius;

(x, y) = photographic coordinates in millimeters. These are affected by errors of refraction.

(l', b') = moon's topocentric librations, or selenographic longitude and latitude of telescope at exposure;

(α, δ) = geocentric right ascension and declination of moon at exposure;

(α', δ') = topocentric right ascension and declination of moon at exposure;

H = geocentric hour angle of moon;

*These methods are described in a general way in Chapter 2, *The Moon, Meteorites and Comets*, eds. B. M. Middlehurst and G. P. Kuiper, 1963.

H' = topocentric hour angle of moon;
Q' = topocentric parallactic angle;
ZD' = topocentric zenith distance, namely, zenith distance from astronomical zenith as affected by parallax;
ZD'' = geocentric zenith distance, or distance of moon from geocentric zenith;
π = moon's parallax interpolated for instant of exposure;
s = moon's tabular semidiameter interpolated for instant of exposure;
s' = moon's topocentric semidiameter;
U = coefficient of refraction in circular measures;
ρ = distance from center of earth to observatory in units of the earth's equatorial radius.

Other notation of a more transient character is indicated in the text.

2. *Interpolation of Astronomical Data for Instant of Exposure*

All time-dependent astronomical variables entering into the reductions are interpolated from the ephemeris with the simple formula

$$u = u_0 + f\Delta + \tfrac{1}{2} f(f-1)\Delta', \qquad (2.1)$$

in which u is the value at exposure, u_0 is the value at the beginning of an ephemeris interval, f is the fraction of the interval, and Δ and Δ' are the first and second differences.

3. *Computation of Topocentric Hour Angle and Declination*

The geocentric hour angle of the moon is computed from

$$\begin{aligned} H = {} & \text{Greenwich sideral time at } 0^{\text{h}} \text{ UT} \\ & + \text{Sideral equivalent of UT} \\ & - \text{W longitude of observatory} \\ & - \text{Geocentric RA of moon.} \end{aligned} \qquad (3.1)$$

The correction ΔH to convert this to the topocentric hour angle H' is found from

$$\tan \Delta H = \frac{\rho \sin \pi \sin H \cos \varphi'}{\cos \delta - \rho \sin \pi \cos H \cos \varphi'}. \qquad (3.2)$$

In this ρ is the geocentric distance of the observatory in units of the earth's equatorial radius, while φ' is the geocentric latitude of the observatory. The topocentric declination is computed directly from

$$\tan \delta' = \frac{(\sin \delta - \rho \sin \pi \sin \varphi') \cos H'}{\cos \delta \cos H - \rho \sin \pi \cos \varphi'}. \qquad (3.3)$$

4. *Computation of Zenith Distances and Parallactic Angles*

A little care is required in connection with zenith distances and parallactic angles. Clearly, the astronomical zenith must be used for the computation of the effects of refraction, while the geocentric zenith is appropriate for the correction of the lunar semidiameter.

If φ is the *astronomical* latitude of the observatory, the topocentric zenith distance is found from

$$\cos ZD' = \sin \varphi \sin \delta' + \cos \varphi \cos \delta \cos H'. \qquad (4.1)$$

The topocentric parallactic angle is then computed from

$$\sin Q' = \sin H' \cos \varphi \operatorname{cosec} ZD' \qquad (4.2)$$

and

$$\cos Q' = \frac{\sin \varphi - \sin \delta' \cos ZD'}{\cos \delta' \sin ZD'}. \qquad (4.3)$$

On the other hand, the distance of the moon from the geocentric zenith is found from

$$\cos ZD'' = \sin \varphi' \sin \delta + \cos \varphi' \cos \delta \cos H. \qquad (4.4)$$

5. *Computation of Topocentric Semidiameter*

The tabular value s of the moon's angular semidiameter is interpolated for the instant of exposure and is then converted to its topocentric value s' by

$$\sin s' = \frac{\sin s}{\sqrt{(1 + \rho^2 \sin^2 \pi - 2\rho \sin \pi \cos ZD'')}} \qquad (5.1)$$

where ZD'' is computed as in (4.4).

6. *Computation of Differential Refractions*

The refraction is conveniently computed from Comstock's formula, which agrees with the Pulkova refractions down to 70° zenith distance. The formula is given as

$$r = (983b \tan ZD') / (460 + F), \qquad (6.1)$$

where

r = astronomical refraction in seconds of arc;
b = barometric pressure in inches;
F = temperature Fahrenheit;
ZD' = topocentric zenith distance.

The refraction coefficient in radians is then

$$U = (983b \sin 1'')/(460 + F), \tag{6.2}$$

and the compression of the photograph by refraction along the vertical circle is

$$1/\kappa = 1 - U \sec^2 ZD'. \tag{6.3}$$

The barely appreciable compression in the horizontal direction is

$$1/\kappa' = 1 - U. \tag{6.4}$$

7. *Computation of the Fictitious Photograph*

The control points with their known rectangular selenodetic coordinates (E, F, G) are used to construct a fictitious photograph to which the real photograph is fitted. This construction in computational form is now detailed. The instantaneous solid coordinates are

$$\left.\begin{aligned} X &= E \cos l' - G \sin l' \\ Y &= -E \sin l' \sin b' + F \cos b' - G \cos l' \sin b' \\ Z &= E \sin l' \cos b' + F \sin b' + G \cos l' \cos b' \end{aligned}\right\}. \tag{7.1}$$

The Y-axis is the projection of the moon's polar axis on the plane of the limb while the Z-axis is directed toward the telescope. The transformation (7.1) is orthogonal so there is no change of scale or of origin. The computation of the topocentric librations l' and b' is described in *Comm. LPL* No. 10.

The solid rectangular coordinates (X, Y, Z) are transformed into plane rectangular coordinates in the plane of the limb by

$$\left.\begin{aligned} X' &= X/(1 - Z \sin s') \\ Y' &= Y/(1 - Z \sin s') \end{aligned}\right\}. \tag{7.2}$$

The point (X', Y') is thus the intersection of the line of sight with the plane of the limb. The aggregate (X', Y') represents a photograph free of refraction with the coordinates expressed in units of the moon's radius. The axis of Y' coincides with that of Y and hence the position of both, measured from the vertical circle through the center of face, is

$$\alpha = C' - Q', \tag{7.3}$$

where C' is the topocentric value of the position angle of the moon's axis, computed as in *Comm.* No. 10. The horizontal and vertical coordinates in the refraction-free fictitious photograph are

$$\left.\begin{aligned} u &= X' \cos \alpha - Y' \sin \alpha \\ v &= Y' \cos \alpha + X' \sin \alpha \end{aligned}\right\}. \tag{7.4}$$

Finally, the horizontal and vertical coordinates, as affected by refraction, are

$$u' = u/\kappa', \quad v' = v/\kappa. \tag{7.5}$$

8. *The Least Squares Solution*

The aggregate (u', v') for the controls defines a picture that is strictly comparable to the real photograph, except for random errors both in the measures and in the control positions. Thus u' and v' are derived from the photographic coordinates x and y by a transformation that takes into account differences of scale, origin, and orientation. The appropriate transformation is

$$\left.\begin{aligned} u' &= \mu (x \cos \theta - y \sin \theta) + d \\ v' &= \mu (y \cos \theta + x \sin \theta) + h \end{aligned}\right\}, \tag{8.1}$$

in which μ is a scale factor, θ represents a rotation, and d and h are shifts. For present purposes, the last is written more conveniently as

$$\left.\begin{aligned} u' &= px - qy + d \\ v' &= py + qx + h \end{aligned}\right\}, \tag{8.2}$$

so that $\quad p = \mu \cos \theta,$
$\quad q = \mu \sin \theta,$

and

$$\mu^2 = p^2 + q^2. \tag{8.3}$$

Application of (8.2) to the n controls leads to the normal equations,

$$\left.\begin{aligned} nd + p\Sigma x - q\Sigma y &= \Sigma u' \\ nh + p\Sigma y + q\Sigma x &= \Sigma v' \\ d\Sigma x + h\Sigma y + p\Sigma (x^2 + y^2) &= \Sigma (u'x + v'y) \\ -d\Sigma x + h\Sigma x + q\Sigma (x^2 + y^2) &= \Sigma (v'x - u'y) \end{aligned}\right\}. \tag{8.4}$$

At this stage the residuals in u' and v' are examined to ensure that there is no misidentification of the controls.

9. *Computation of the Selenographic Coordinates*

The standard direction-cosines (ξ, η, ζ) of the line-of-sight projection of the selenodetic point on the mean spherical datum are computed merely to identify the point. These values vary slightly from photograph to photograph because of differences in the optical librations. Unfortunately, they also vary because of errors in the control points and the measures. The required values (ξ, η, ζ) are easily obtained by reversing the steps used to obtain the fictitious photograph.

First, the values (u', v') are computed from (8.2).

The refraction-free horizontal and vertical coordinates are then found by inverting (7.5), i.e.,

$$u = u'\kappa' \quad \text{and} \quad v = v'\kappa. \qquad (9.1)$$

Next, inverting (7.4),

$$\left.\begin{aligned} X' &= u\cos\alpha + v\sin\alpha \\ Y' &= v\cos\alpha - u\sin\alpha \end{aligned}\right\}. \qquad (9.2)$$

Equations (7.2) now have to be solved for *X, Y* and *Z;* note that in this case

$$X^2 + Y^2 + Z^2 = 1, \qquad (9.3)$$

since the required point lies on the mean sphere. An iterative solution is possible but a more direct approach seems preferable. Equations (7.2) can be written as

$$\left.\begin{aligned} X &= X'(1 - Z\sin s') \\ Y &= Y'(1 - Z\sin s') \end{aligned}\right\}. \qquad (9.4)$$

Substituting these in (9.3) and solving the quadratic for *Z*, we have

$$Z = \frac{T\sin s' + \surd[1 - T(1 - sin^2 s')]}{1 + T\sin^2 s'}, \qquad (9.5)$$

where

$$T = X'^2 + Y'^2.$$

The remaining values are then found from (9.4). Finally the standard direction-cosines are computed by inverting (7.1), i.e.,

$$\left.\begin{aligned} \xi &= X\cos l' - Y\sin l'\sin b' + Z\sin l'\cos b' \\ \eta &= Y\cos b' + Z\sin b' \\ \zeta &= -X\sin l' - Y\cos l'\sin b' + Z\cos l'\cos b' \end{aligned}\right\}. \qquad (9.6)$$

10. Computation of Refraction-Free Photographic Coordinates

The primary aim of the reductions at this stage is the derivation of refraction-free photographic coordinates. Indeed these are obtained as X' and Y' in (9.2), but in units of the moon's radius. They are not acceptable in this form since they contain a factor that may be in error because of errors in the controls, and in the librations also. However, referring to (8.3) one can see that the factor for converting millimeters to units of the moon's radius is

$$\mu = \surd(p^2 + q^2). \qquad (10.1)$$

Hence, the appropriate factor for converting X' and Y' to millimeters is $1 / \mu$, and the refraction-free photographic coordinates are

$$\left.\begin{aligned} x' &= X' / \mu \\ y' &= Y' / \mu \end{aligned}\right\}. \qquad (10.2)$$

These are virtually free from the effects of errors of the controls and librations, although they may be affected by error in the computed refraction. Note that the origin of (x', y') is very close to the true center of face, and that the y'-axis coincides closely with the projection of the moon's polar axis on the plane of the limb.

Acknowledgments. This research has been supported by the Air Force Cambridge Research Laboratories, U.S. Air Force, under Contract AF19 (628)–4332.

REFERENCES

Arthur, D. W. G. 1963, "Selenography," *The Moon, Meteorites and Comets,* eds. B. M. Middlehurst and G. P. Kuiper (Chicago: University of Chicago Press), Chap. 2.

Arthur, D. W. G., and Huzzen, C. S. 1962, "Topocentric Librations of the Yerkes Lunar Photographs," *Comm. LPL,* **1**, (10), 45.

Breece, S., Hardy, M., and Marchant, M. Q. 1964, *Horizontal and Vertical Control for Lunar Mapping,* Army Map Service Tech. Rept. No. 29, Pt. II.

Meyer, D. L., and Ruffin, B. W. 1965, *Coordinates of Lunar Features, Group I and II Solutions,* ACIC Tech. Paper No. 15, Aeronautical Chart and Information Center, U.S. Air Force, St. Louis, Mo.

Schrutka-Rechtenstamm, G. 1958, "Neureduktion der 150 Mondpunkte der Breslauer Messungen von J. Franz," *Österr. Akad. Wiss. Math.-Naturw. Kl. Sitzber.,* Abt. II, Vol. 167.