NO. 62 THE COMPUTATION OF SELENODETIC COORDINATES USING THE LIBRATIONS

by D. W. G. ARTHUR

August 3, 1965

ABSTRACT

Details are given of the LPL method for combining measures on different lunar photographs, using the librations, to compute solid selenodetic coordinates. Published methods are critically examined, and the correct treatment of pseudo-observations is outlined. A first-order treatment of the error theory for selenodetic positions is derived from the LPL method of combining plates.

1. Introduction

Selenodetic coordinates have been computed from measures on lunar photographs by Franz (1899), Saunder (1905), Baldwin (1963), Schrutka-Rechtenstamm (1958), the U.S. Army Map Service (1964), and the Aeronautical Chart and Information Center of the U.S. Air Force (1965). With the exception of Baldwin, each of these authors uses a different method for converting the measures into selenodetic coordinates; since Baldwin uses the same method as Saunder, his computations do not require further discussion. Goudas (1965) has published a method of computation in which the mathematics are in error.

All these authors start from the same base, namely, plane coordinate measures on photographs with different optical librations. This is the so-called stereoscopic method, although the name is a misnomer, since stereoscopy is not generally used in the method and certainly is not essential. The methods used are often characterized by considerable indirectness of approach. Since reliable selenodetic coordinates can be obtained only with considerable overdetermination in the measures, all investigators have, in their reductions, used the method of least squares; but, with a single exception, the method has not been correctly applied. The resulting selenodetic coordinates are *not* least squares values and their precision is not correctly estimated. The method described here is direct and rigorous in its use of least squares. It cannot be very different in principle from that of the Army Map Service (1960), but the exposition given below is probably simpler.

The derivations of selenodetic coordinates from measures on lunar photographs may be classified as follows, when regarded from the viewpoint of least squares:

I. Schemes in which the observation equations are stated in terms of the measured quantities;

II. Schemes in which the observation equations are stated in terms of pseudo-observations, namely, transforms of the original measures.

The second class may be illustrated by Saunder's method. From each single photograph Saunder de-

rived the standard direction-cosines (ξ, η, ζ) of the line-of-sight projection of the selenodetic point on the mean lunar spherical datum. His pseudo-observations were:

$$\delta\xi = \xi - \bar{\xi},$$
$$\delta\eta = \eta - \bar{\eta},$$

where $\bar{\xi}$ and $\bar{\eta}$ are the values of ξ and η meaned over all plates. Saunder then treated $\delta\xi$ and $\delta\eta$ as independent measurements, and in doing so made a mistake that unfortunately has been much too common in selenodetic investigations.

Schemes of the second class are very convenient if the investigator applies the method of least squares in a mechanical way and ignores the correlations between the pseudo-observations. However, if these correlations are taken into account, and the least squares techniques are made rigorous, these schemes lose their advantages and become extremely laborious. This point is made clear below.

The method proposed in this paper belongs to the first class. However, since the use of pseudo-observations is so widespread, it seems appropriate to sketch the correct method of applying the method of least squares to correlated data. It is important that the investigator determine whether his observation equations are stated in terms of independent measurements, or in terms of pseudo-observations that are transforms of these.

2. *The Principle of Least Squares for Correlated Observations*

The following notation is used:*

x = column matrix of unknown parameters $x_1, x_2, \ldots x_n$ to be determined from indirect measures;

l = column matrix of measured quantities $l_1, l_2, \ldots l_m$;

A = rectangular matrix of coefficients in the observation equations. A has m rows and n columns.

v = column matrix of residuals $v_1, v_2, \ldots v_m$;

P = diagonal weight matrix with the weights $p_1, p_2, \ldots p_m$ as the elements of the leading diagonal. All other elements of P are zero.

Other notation will be introduced as required, but boldface lowercase letters will always represent column matrices.

*Matrix notation is used for compactness of exposition.

When the observations are independent and the errors are normally distributed, the most probable values of the unknowns follow from the minimization of the diagonal quadratic form:

$$\boldsymbol{v}^T\boldsymbol{P}\boldsymbol{v} = \Sigma pv^2 \tag{2.1}$$

where $\boldsymbol{v}^T$ is the transpose of $\boldsymbol{v}$. When applied to the indirect observations

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{l}, \tag{2.2}$$

this minimization leads to the normal equations

$$\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{l}, \tag{2.3}$$

which uniquely determine the most probable value of the unknown parameters $x_1, x_2, \ldots x_n$. The variance of an observation of unit weight is estimated from

$$\sigma^2 = \Sigma pv^2 / (m - n). \tag{2.4}$$

The above results are well known, but evidently it is not so well known that they are valid only when there is no correlation between the observed quantities $l_1, l_2, \ldots l_m$.

The discussion of correlation given here is restricted to algebraic correlation — that is, correlation generated by transformation of the variables. Let $\tau_1, \tau_2, \ldots \tau_n$ represent a set of observations, and let c_{ik} represent the covariance between τ_i and τ_k. Then the entire set of variances and covariances is represented by the *covariance matrix:*

$$C_{\tau\tau} = \begin{pmatrix} c_{11}, c_{12}, \ldots c_{1n} \\ c_{21}, c_{22}, \ldots c_{2n} \\ c_{n1}, c_{n2}, \ldots c_{nn} \end{pmatrix}. \tag{2.5}$$

This is square and symmetric, and its diagonal elements represent the variances. For independent variables the off-diagonal elements are zero.

Now let a second set $(y_1, y_2, \ldots y_m)$ be derived from the τ_i by the transformation

$$y_i = f_i(\tau_1, \tau_2, \ldots \tau_n) \qquad i = 1, 2, \ldots m, \tag{2.6}$$

and write

$$f_{ik} = \frac{\partial y_i}{\partial \tau_k},$$

so that we have the matrix relation

$$dy = \boldsymbol{F}d\tau,$$

where f_{ik} is the typical element of the matrix $\boldsymbol{F}$. Then, it may be shown that the covariance matrix for the y_i is

$$\boldsymbol{C}_{yy} = \boldsymbol{F}\boldsymbol{C}_{\tau\tau}\boldsymbol{F}^T. \tag{2.7}$$

In general, even when the set $(\tau_1, \tau_2, \ldots \tau_n)$ is correlation-free, the set $(y_1, y_2, \ldots y_m)$ is not. In other words, transformation of an independent set will generally produce a correlated set.

In adjustment theory, it is usually convenient to replace the covariance matrix C by the *correlation matrix* Q where

$$Q = \frac{1}{\sigma^2} C, \tag{2.8}$$

and, as before, σ^2 is the variance of an observation of unit weight. The reciprocal of Q is called the *generalized weight matrix* G, i.e.,

$$G = Q^{-1}. \tag{2.9}$$

The matrix G reduces to P when the measures are uncorrelated, and indeed G is merely a generalization of the weight matrix and plays the same role for correlated measures that P does for independent measures.

For correlated measures, the quality to be minimized is the symmetric quadratic form

$$v^T G v = \Sigma g_{ik} v_i v_k, \tag{2.10}$$

and for indirect measures this minimization leads to the normal equations

$$A^T G A x = A^T G l. \tag{2.11}$$

The variance of an observation of unit weight is estimated using

$$\sigma^2 = \frac{\Sigma g_{ik} v_i v_k}{m - n}. \tag{2.12}$$

The last three results are obvious generalizations of (2.1), (2.3), and (2.4). If the pseudo-observations are derived by applying the matrix F to independent observations with weight matrix P, then the correlation matrix of the pseudo-observations is clearly

$$Q = F P^{-1} F^T,$$

and the generalized weight matrix must be computed as the reciprocal of Q. This inversion is in addition to that occurring in the solution of the normal equations, and may be a serious drawback to the use of pseudo-observations in a particular case.

One other result should be noted. When the observations are independent, the inverse normal matrix is also the correlation matrix of the unknown parameters, i.e.,

$$Q_{xx} = (A^T P A)^{-1}. \tag{2.13}$$

In the case of correlated observations, the inverse normal matrix is still the correlation matrix of the unknown parameters x_i, so that in this case (2.13) is generalized to

$$Q_{xx} = (A^T G A)^{-1}. \tag{2.14}$$

The above results are all that are required for the case of indirect observations subject to correlation of an algebraic character. For a more general discussion and derivation of the results, the reader is referred to the works of Tienstra (1956) or Linnik (1961).

3. *Remarks about Published Computation Schemes*

An examination of the computation schemes of Franz, Pickering, Saunder, and Schrutka-Rechtenstamm shows that these are all phrased in terms of pseudo-observations, and the same is true of the recently published method of the Aeronautical Chart and Information Center. Various types of pseudo-observations are used, but *these are all error-correlated*. It follows that the use of (2.3) and (2.4) is not valid, although, in fact, these were used; hence, the values obtained are not least squares values, and the accuracy of the estimates of the precision is compromised.

The scheme proposed by Goudas requires separate comment since his final results, equations (30) and (31), may not be correct. The Goudas equation,

$$\delta \mathbf{q} = (\mathbf{t} + \delta \mathbf{p}) / \mathbf{p},$$

cannot be correct, since the vectors $\delta \mathbf{q}$ and $\mathbf{p}$ are collinear according to his definitions on page 191 of his paper, and can only differ by a scalar factor. Accordingly, his equations (14), (17), and (18) are wrong. I have not traced his formulation further since his entire scheme is cumbersome from the viewpoint of the programmer.

4. *The LPL Method for the Computation of Selenodetic Coordinates*

As noted above, the computation schemes already published are characterized by a certain indirectness of approach. Let (E, F, G) be the provisional values of the solid rectangular coordinates of the selenodetic points. These are assumed to be in units of the moon's radius. Let (x_0, y_0) be the observed coordinates in the plane of the photograph. These are assumed to be free of refraction. If (x_c, y_c) are the photographic coordinates computed from the provisional values (E, F, G), then the observation equations for a single observation on one plate are

$$\left.\begin{aligned}\frac{\partial x}{\partial E}\cdot\delta E+\frac{\partial x}{\partial F}\cdot\delta F+\frac{\partial x}{\partial G}\cdot\delta G&=x_0-x_c\\ \frac{\partial y}{\partial E}\cdot\delta E+\frac{\partial y}{\partial F}\cdot\delta F+\frac{\partial y}{\partial G}\cdot\delta G&=y_0-y_c\end{aligned}\right\},\quad(4.1)$$

in which δE, δF, and δG are the required corrections to the provisional values. Since close approximations to E, F, and G can be computed from measures on one plate, one iteration is sufficient, unless the computations indicate large errors in the controls.

The values E, F, and G are transformed to photographic coordinates as follows. First, the instantaneous coordinates X, Y, and Z are computed from

$$\left.\begin{aligned}X&=aE+bF+cG\\ Y&=eE+fF+gG\\ Z&=iE+jF+kG\end{aligned}\right\}.\quad(4.2)$$

The coefficients of this rotational transformation are computed from the topocentric librations l' and b',

$$\begin{aligned}\text{i.e.,}\quad a&=\cos l',\\ b&=0,\\ c&=-\sin l',\\ e&=-\sin l'\sin b',\\ f&=\cos b',\\ g&=-\cos l'\sin b',\\ i&=\sin l'\cos b',\\ j&=\sin b',\\ k&=\cos l'\cos b'.\end{aligned}$$

The corresponding coordinates of the line-of-sight projection of the point in the plane of the limb are:

$$\left.\begin{aligned}X'&=X/(1-Z\sin s')\\ Y'&=Y/(1-Z\sin s')\end{aligned}\right\},\quad(4.3)$$

where s' is the augmented lunar semidiameter. Since the photographic coordinates are assumed to be free of refraction, these are connected to X' and Y' by relations of the type

$$\left.\begin{aligned}x&=pX'-qY'+d\\ y&=pY'+qX'+h\end{aligned}\right\}.\quad(4.4)$$

Applying (4.4) to the controls, that is, to points whose selenodetic values are known, the resulting normals for n such controls are:

$$\left.\begin{aligned}nd+\ p\Sigma X'-q\Sigma Y'&=\Sigma x\\ nh+\ p\Sigma Y'+q\Sigma X'&=\Sigma y\\ d\Sigma X'+h\Sigma Y'+p\Sigma(X'^2+Y'^2)&=\Sigma(xX'+yY')\\ -d\Sigma Y'+h\Sigma X'+q\Sigma(X'^2+Y'^2)&=\Sigma(yX'-xY')\end{aligned}\right\}.\quad(4.5)$$

Solving these for the plate constants p, q, d, and h, then for *all* points on the plate, (4.4) gives

$$\left.\begin{aligned}x_c&=pX'-qY'+d\\ y_c&=pY'+qX'+h\end{aligned}\right\}.\quad(4.6)$$

At this point it may be remarked that, even if the controls are errorless (which is certainly not true), the observations on these are not errorless; hence p, q, d, and h will be error-correlated. For reasons of tractability, this correlation must be ignored. However, it should be noted that, in principle, x_c and y_c are error-correlated even if this correlation is unimportant.

The above scheme appears to be particularly convenient when (4.2), (4.3), and (4.6) are combined as a single subroutine in the computations, for then by computing the lattice

$$\begin{aligned}&(E,F,G),\\ &(E+.001,F,G),\\ &(E,F+.001,G),\\ &(E,F,G+.001),\end{aligned}$$

the formal derivation of the partial derivatives in (4.1) is avoided. If the corresponding photographic values are:

$$\begin{aligned}&(x_c,y_c),\\ &(x_e,y_e),\\ &(x_f,y_f),\\ &(x_g,y_g),\end{aligned}$$

then the partial derivatives are computed as

$$\begin{aligned}\frac{\partial x}{\partial E}&=(x_e-x_c)/.001,\\ \frac{\partial x}{\partial F}&=(x_f-x_c)/.001,\\ \frac{\partial x}{\partial G}&=(x_g-x_c)/.001,\\ \frac{\partial y}{\partial E}&=(y_e-y_c)/.001,\qquad\text{etc.}\end{aligned}$$

However, for those who prefer the formal expressions, the derivatives are easily found by approximating (4.3) to

$$\begin{aligned}X'&=X(1+Z\sin s'),\\ Y'&=Y(1+Z\sin s'),\end{aligned}$$

and then using identities of the type

$$\frac{\partial x}{\partial E}=\frac{\partial x}{\partial X'}\frac{\partial X'}{\partial E}+\frac{\partial x}{\partial Y'}\frac{\partial Y'}{\partial E},\qquad\text{etc.}$$

These lead to the computing scheme;

$$S = 1 + Z \sin s', \tag{4.7}$$

$$\left.\begin{aligned} a' &= aS + iX \sin s' \\ b' &= bS + jX \sin s' \\ c' &= cS + kX \sin s' \\ e' &= eS + iY \sin s' \\ f' &= fS + jY \sin s' \\ g' &= gS + kY \sin s' \end{aligned}\right\}, \tag{4.8}$$

$$\left.\begin{aligned} \frac{\partial x}{\partial E} &= pa' - qe' \\ \frac{\partial x}{\partial F} &= pb' - qf' \\ \frac{\partial x}{\partial G} &= pc' - qg' \end{aligned}\right\}, \tag{4.9}$$

$$\left.\begin{aligned} \frac{\partial y}{\partial E} &= qa' + pe' \\ \frac{\partial y}{\partial F} &= qb' + pf' \\ \frac{\partial y}{\partial G} &= qc' + pg' \end{aligned}\right\}. \tag{4.10}$$

Each point observed on each plate gives two observation equations leading to 3×3 normals for each point. If the precision of the absolute altitude is to be estimated, it is advisable to compute the reciprocal normal matrix for each point.

If the corrections δE, δF, or δG are appreciable for any of the controls, it may be advisable to recompute the plate constants p, q, d, and h with the corrected values. In any case, the residuals of the observation equations (4.1) are likely to reveal at least one control for which the measures are discordant, so that a second solution for the plate constants is almost certainly necessary.

5. *Estimates of Precision*

The precision of the measures can be estimated in a formal way; but even with a dozen or so photographs the sample is still rather small, and it is difficult to distinguish real variations in precision from statistical fluctuations. The precisions of x and y may be quite different on the same plate; and moreover, the mean precision of x and y will vary for the same point from plate to plate, depending on phase and seeing. A complete analysis of variance for the measures should therefore take into account three sources of variance: (i) the point, (ii) the plate, and (iii) the phase.

Furthermore, it seems unlikely that the x and y errors are entirely uncorrelated, particularly in limb regions where the craters appear as elongated ellipses. When these remarks are borne in mind, it will be seen that any conclusions about precision must be regarded with some caution. However, for the purposes of making some kind of estimate of the precision of the measures, the selenodetic coordinates, and the absolute altitudes, the following analysis is provided.

Let it be assumed that the measures x and y are of the same variance and are pairwise independent. Let the elements of the inverse normal matrix for one point be, in the usual abbreviated form,

$$\begin{array}{lll} r_{11}, & r_{12}, & r_{13}, \\ & r_{22}, & r_{23}, \\ & & r_{33}. \end{array}$$

The variance of the measures is computed from

$$\sigma^2 = \Sigma v^2/(n-3), \tag{5.1}$$

where n is the number of observation equations and v is a typical residual. The standard errors of the selenodetic coordinates of the point are estimated from

$$\left.\begin{aligned} \sigma_E &= \sigma\sqrt{r_{11}} \\ \sigma_F &= \sigma\sqrt{r_{22}} \\ \sigma_G &= \sigma\sqrt{r_{33}} \end{aligned}\right\}. \tag{5.2}$$

The covariances are

$$\begin{aligned} c_{EF} &= r_{12}\sigma^2, \\ c_{EG} &= r_{13}\sigma^2, \\ c_{FG} &= r_{23}\sigma^2. \end{aligned}$$

The absolute altitude h is computed from

$$h = \sqrt{(E^2 + F^2 + G^2)} - 1, \tag{5.3}$$

whence

$$dh \approx EdE + FdF + GdG. \tag{5.4}$$

Applying (2.7), the variance of the absolute altitude comes to

$$\begin{aligned} \sigma_h^2 = {} & \sigma^2 (E^2 r_{11} + F^2 r_{22} + G^2 r_{33} \\ & + 2EFr_{12} + 2EGr_{13} + 2FGr_{23}). \end{aligned} \tag{5.5}$$

6. *Dispersion in Selenodetic Coordinates as a Function of Position on the Disk*

The dispersion in the selenodetic coordinates may vary as a function of position on the lunar surface for two reasons: (i) for reasons of a geometric nature such that even when the measures are made with uniform precision over the entire disk, points

in some positions are nevertheless determined with more precision than points in other positions; (ii) because of a gradual shift in the nature of the measured point from center of face to limb, the usual assumption being that as the limb is approached the measures are restricted to larger and larger craters with a consequent decrease in precision.

It will now be shown that when the measures are made with uniform precision over the entire disk, then the selenodetic coordinates also are determined with uniform precision, so that assumption (i) is not true.

To simplify the problem, one should note the very close relations between the X and Y of this paper and the photographic coordinates x and y. For the purposes of a first-order theory, it may be assumed that the transformation (4.4) is errorless. Then, since this transformation is merely an orthogonal transformation compounded with a change of origin and scale, the assumption that x and y are independent and of the same variance implies that X' and Y' are also independent with the same variance. Thus, since these do not differ much from X and Y, it is valid to treat X and Y as the independent observed quantities and to assume that they have the same variance. Let this variance be μ^2.

The observation equations for a single point can now be simplified to

$$\left.\begin{aligned} aE + bF + cG &= X_0 - X_c \\ eE + fF + gG &= Y_0 - Y_c \end{aligned}\right\}. \qquad (6.1)$$

The normal matrix, in abbreviated form is

$$\begin{array}{lll} \Sigma(a^2+e^2), & \Sigma(ab+ef), & \Sigma(ac+eg), \\ & \Sigma(b^2+f^2), & \Sigma(bc+fg), \\ & & \Sigma(c^2+g^2). \end{array} \qquad (6.2)$$

Without going any further it may be noted that the elements of this matrix do *not* contain the selenodetic positions, and hence the elements of the inverse normal matrix are also independent of the position of the point. But the diagonal elements of the inverse matrix are also the variances of the selenodetic coordinates (E, F, G) expressed in units of the variance of an observation of unit weight of X or Y.

Hence the result: *when the observations are made with uniform precision over the disk, then the dispersion in the selenodetic coordinates (E, F, G) is represented by an error-ellipsoid whose dimensions are independent of position on the disk.*

The analysis may be carried a stage further by noting that

$$\begin{aligned} a^2 + e^2 + i^2 &= 1, \\ b^2 + f^2 + g^2 &= 1, \\ c^2 + g^2 + k^2 &= 1, \\ ab + ef + ij &= 0, \\ ac + eg + ik &= 0, \\ bc + fg + jk &= 0. \end{aligned}$$

Using these, one may write the normal matrix (for m plates)

$$\begin{array}{lll} m - \Sigma i^2, & -\Sigma ij, & -\Sigma ik \\ & m - \Sigma j^2, & -\Sigma jk \\ & & m - \Sigma k^2. \end{array}$$

The summations are over all plates used to determine the point. Expanding i, j, and k in terms of the topocentric librations l' and b' and dropping the primes for convenience, we have, after approximating the sines to the angles and the cosines to the second order in the angles,

$$\begin{array}{lll} m - \Sigma l^2, & -\Sigma lb, & -\Sigma l, \\ & m - \Sigma b^2, & -\Sigma b, \\ & & \Sigma(l^2 + b^2). \end{array}$$

In this all terms of the third order and higher have been omitted. Assume now that the set of measured plates is well balanced in librations, so that

$$\Sigma l \approx 0, \quad \Sigma b \approx 0.$$

Then the normal matrix reduces to

$$\begin{array}{lll} m - \Sigma l^2, & -\Sigma lb, & 0, \\ & m - \Sigma b^2, & 0, \\ & & \Sigma(l^2 + b^2). \end{array}$$

The determinant of this matrix is

$$\Delta = \Sigma(l^2 + b^2)\,[(m - \Sigma l^2)(m - \Sigma b^2) - (\Sigma lb)^2].$$

Now at their maxima l and b are about 0.2 so this determinant may be approximated to

$$\Delta \approx m^2\,\Sigma(l^2 + b^2). \qquad (6.3)$$

Applying Cramer's rule, the diagonal elements of the inverse normal matrix compute to

$$r_{11} = \frac{(m - \Sigma b^2)\,\Sigma(l^2 + b^2)}{m^2\,\Sigma(l^2 + b^2)} \approx 1/m,$$

$$r_{22} = \frac{(m - \Sigma l^2)\,\Sigma(l^2 + b^2)}{m^2\,\Sigma(l^2 + b^2)} \approx 1/m,$$

$$r_{33} = \frac{(m - \Sigma l^2)(m - \Sigma b^2)}{m^2\,\Sigma(l^2 + b^2)} \approx 1/\Sigma(l^2 + b^2).$$

Hence, the standard errors of the selenodetic coordinates are

$$\left.\begin{aligned}\sigma_E &= \mu/\sqrt{m}\\ \sigma_F &= \mu/\sqrt{m}\\ \sigma_G &= \mu/\sqrt{\Sigma(l^2+b^2)}\end{aligned}\right\}, \qquad (6.4)$$

where μ is the standard error of X and Y in a single determination, while l and b are the topocentric librations. For each plate,

$$l^2 + b^2 < 0.08,$$

so that $\sigma_G^2/\sigma_E^2 = \sigma_G^2/\sigma_F^2 \geqslant 12.$

That is, the standard error of the earthward-directed coordinate G is at least 3.5 times greater than the standard error of either E or F.

Thus, with the assumptions made above (concerning homogeneity in the measures), the precision of the absolute altitudes near the limb is at least 3.5 times better than in the area at the center of the disk.

All the above depends on the assumption that the measures are made with uniform precision, whereas some writers have assumed that this precision falls off toward the limb. This assumption, in turn, depends on two others, which are (i) that the difficulities of measuring small craters increase with distance from the center of face, and (ii) that the positions of large craters are determined with less precision. One of these questions can be settled by an examination of existing selenodetic positions. The 150 points of Franz (and therefore of Schrutka-Rechtenstamm) are not very useful for this enquiry because the Lick photographs measured by the former were quite lacking in resolution. Also, Franz selected points that are unsuitable for selenodetic purposes. The numerous points used by Baldwin would constitute a much better basis, but, unfortunately, his results are printed in a form that does not permit a ready correlation between position and diameter. The AMS points must also be rejected because a relatively large percentage of these is, in my opinion, unsuitable for selenodetic purposes. The only large body of data suitable for analysis is the catalog of Saunder (1910). The entries in this have been plotted to give a regression of diameter on ζ, where ζ is the third standard direction-cosine. The lower edge of this regression shows virtually no slope between $\zeta = +1.0$ and $\zeta = +0.3$, which implies that from the center of face to 70° from the center of face, there are no particular difficulties in measuring small distinct craters. However, the analysis also shows that as we leave the central regions, the mean crater diameter increases. Figure 2 shows both the mean diameter of the measured craters and the mode, that is, the most commonly occurring diameter for a given value of ζ.

It must be admitted then that Saunder's results show that he tended to measure larger and larger objects as he went from center of face to limb, the overall increase in the mean being about 100 percent. It does *not* follow, however, that the measures of the larger craters are notably less accurate than those for small craters.

A more important factor perhaps in phase photographs is the difficulty of identifying the centers of craters in the equatorial areas near the east and west limbs, since the lighting conditions do not favor the precise delineation of the north and south ends of the craters in these areas.

The topic of the dispersion in selenodetic coordinates as a function of position was discussed by Pickering (1903) who concluded that absolute lunar altitudes are best determined in the central regions of the disk. This view was contradicted by Saunder (1905). Less definite opinions were held by Hopmann (1963) and Baldwin (1963). More recently Goudas has reached the same conclusion as Pickering. The above analysis shows that Saunder's position was correct and the others wrong. Even if we allow a decrease by a factor of 2 in the precision of the measures from center to limb, altitudes of points in the limb regions are still better determined than those near the center of face.

Acknowledgments. Work reported here was supported by the Air Force Cambridge Research Laboratories, U.S. Air Force, under Contract AF19 (628)–4332.

REFERENCES

Baldwin, R. B. 1963, *The Measure of the Moon* (Chicago: University of Chicago Press).

Breece, S., Hardy, M., and Marchant, M. Q. 1964, *Horizontal and Vertical Control for Lunar Mapping,* Army Map Service Tech. Rept. No. 29, Pt. II.

Franz, J. 1899, "Die Figur des Mondes," *Astron. Beobachtungen Königsberg,* Vol. 38.

Goudas, C. L. 1965, "The Selenodetic Control System of the U.S. Army Map Service," *Icarus,* 4, 188–206.

Hopmann, J. 1963, "Ermittelung von Höhen auf dem Monde," *Österr. Akad. Wiss. Math.-Naturw. Kl. Sitzber.,* Abt. II, Vol. 172.

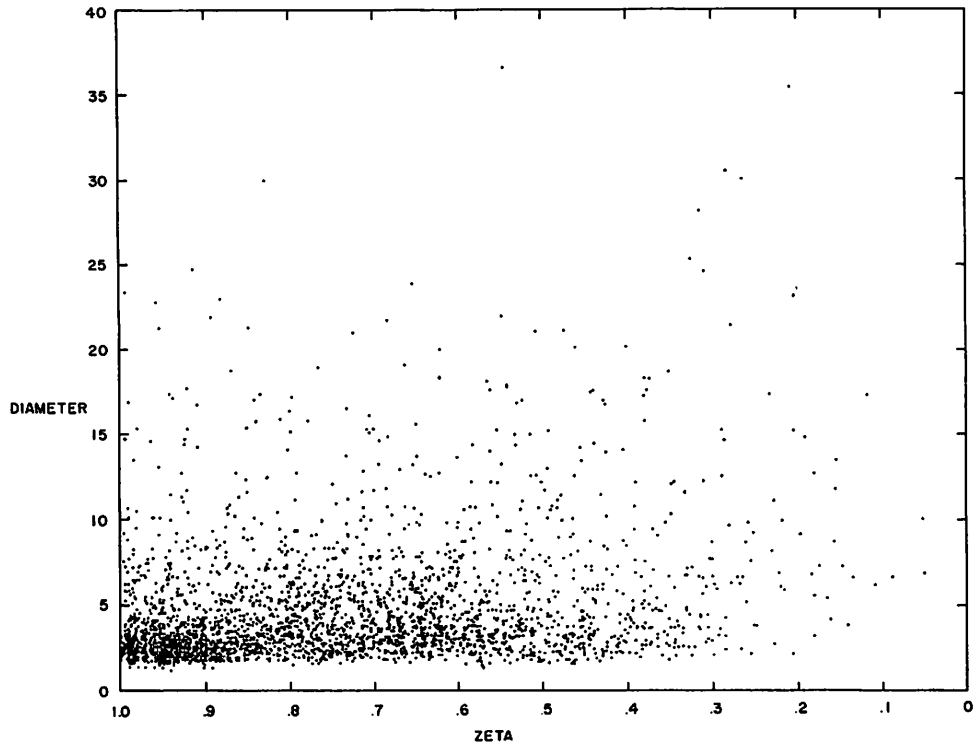

Fig. 1 Saunder points: Diameter versus distance from limb.

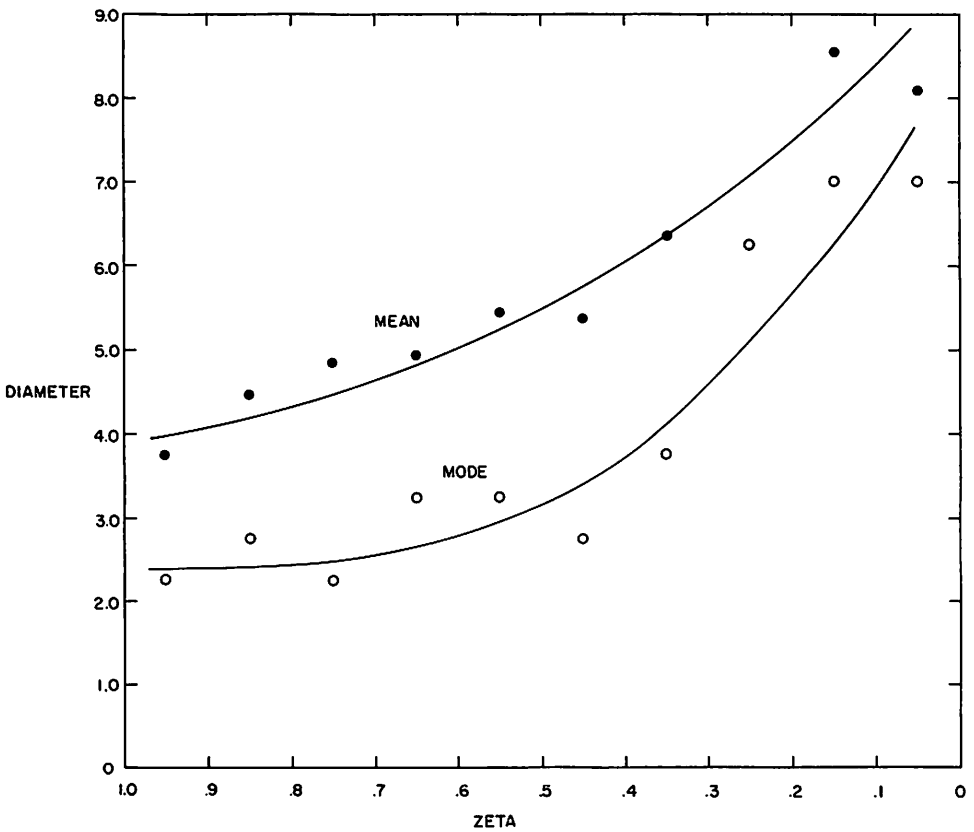

Fig. 2 Saunder points: Diameter versus distance from limb.

REFERENCES — *Continued*

Linnik, Yu. V. 1961, *Method of Least Squares and the Principles of the Theory of Observations* (London: Pergamon Press).

Marchant, M. Q. 1960, *Horizontal and Vertical Control for Lunar Mapping,* Army Map Service Tech. Rept. No. 29, Pt. I.

Meyer, D. L., and Ruffin, B. W. 1965, *Coordinate of Lunar Features, Group I and II Solutions,* ACIC Tech. Paper No. 15, Aeronautical Chart and Information Center, U.S. Air Force, St. Louis, Mo.

Pickering, W. H. 1903, "Absolute Altitudes and Positions on the Moon," *Howard Annals,* Vol. 51.

Saunder, S. A. 1905, "First Attempt to Determine the Figure of the Moon," *Monthly Notices Royal Astronomical Society,* Vol. 65.

————. 1910, "Results of the Measurement of Two Yerkes Negatives," *Memoirs Royal Astronomical Society,* Vol. 60.

Schrutka-Rechtenstamm, G. 1958, "Neureduktion der 150 Mondpunkte der Breslauer Messungen von J. Franz," *Österr. Akad. Wiss. Math.-Naturw. Kl. Sitzber.,* Abt. II, Vol. 167.

Tienstra, J. M. 1956, *Theory of the Adjustment of Normally Distributed Observations* (Amsterdam: Argus).

TABLE OF CONTENTS