NO. 127 A COMPARATOR METHOD FOR DETERMINING RELATIVE LUNAR ALTITUDES

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ABSTRACT

The paper presents details of the direction-cosine method for the determination of relative lunar altitudes from shadow measures on earth-based lunar photographs.

1. Introduction

This paper gives the special mathematics and computing methods for the determination of relative lunar altitudes from measures on lunar photographs. Z. Kopal (Kopal *et al.* 1960) has written on the same topic, but there is surprisingly little in common between his exposition and my own. The principal difference arises from the mathematical language itself. Kopal chose to develop the computations in terms of the formulas of spherical trigonometry. I regard these as appropriate for the reduction of telescopic measures, but they are not particularly efficient for the processing of measures on photographs. The latter are effectively made at one instant and, thus, all the data for the photograph can be represented by a relatively small number of plate constants. It may turn out, however, that the direction-cosine method has no special advantage for photography from space vehicles, and that the more general approach of Kopal will prove more useful for this.

The other differences come from the measuring techniques. Kopal's exposition relates to the microdensitometer measures of photographs with very small fields. However, the portion of the lunar surface registered in each of the photographs used at Manchester is too small to permit the use of standard points for the orientation of the measuring system to the photograph, or to allow the determination of the selenographic positions of the peaks. Thus a discussion of these topics would have been irrelevant. In contrast, the method developed here relates to comparator measures on photographs of larger format. The photographs of the LPL Catalina 61-in. reflector, which cover areas a little larger than the fields of *Photographic Lunar Atlas* (Kuiper *et al.* 1960), are the kind of photographs I have in mind. Each photograph includes sufficient controls to permit a rigorously accurate orientation of the photograph in the comparator. Also, the comparator measures themselves determine the selenographic coordinates of the peaks, thus eliminating one of the principal drawbacks of the Manchester methods.

The effects of approximations at each stage are discussed. Although Kopal claimed completeness in this, he failed to examine the effects of a deviation of the telescope axis from the direction to the center of face. For the Manchester measures these errors are negligible.

The LPL measures are made with a normal two-screw comparator reading to 1 micron in each coordinate. Since the stage has a divided circle reading to 20″, the orientation of the photograph can be set with an error of less than 1 minute of arc. Clearly the comparator pointings at the ends of the shadows are less precise and more subjective than the microdensitometer records, with the result that the comparator method is not particularly precise for short shadows. However, it is always considerably more rapid; the complete comparator readings for one shadow and the position of the peak are obtained in less than one minute.

Each method thus has its strong points and its drawbacks. The microdensitometer method is usually more efficient for short shadows, for which positional and orientation errors have the least effect. The comparator method is usually more efficient for long shadows when the errors of the shadow measures are least effective and the positional errors have the greatest effect.

2. *The Coordinates*

If λ and β are the selenographic longitude and latitude of the peak, its standard direction-cosines are

$$\left.\begin{aligned} \xi &= \cos\beta\sin\lambda \\ \eta &= \sin\beta \\ \zeta &= \cos\beta\cos\lambda \end{aligned}\right\}. \tag{1}$$

These should not be confused with the rectangular coordinates, for which we reserve the notation (E, F, G). If the absolute altitude H and the coordinates are both in units of the moon's radius, then

$$\left.\begin{aligned} E &= (1+H)\xi = (1+H)\cos\beta\sin\lambda \\ F &= (1+H)\eta = (1+H)\sin\beta \\ G &= (1+H)\zeta = (1+H)\cos\beta\cos\lambda \end{aligned}\right\}. \tag{2}$$

A sharp distinction must be made here between the true (ξ, η, ζ) of the peak, and the corresponding values for its straight-line projection on the mean sphere. The true values are not known, whereas the values for the projected point are known with some precision for the standard points and are listed in *Comm.* No. 11 (Arthur 1962). The differences in ξ, η, and ζ between the true and projected values depend on the absolute altitude H and the distance from the center of face. Since H is generally not known, it should be realized that the ξ, η, ζ values derived either from maps or from measures on single photographs tend to be increasingly unreliable as we leave the central regions of the disk. It is doubtful if reliable relative altitudes can be derived at all outside the limits of $\lambda = \pm 30°$ because of the limitations of the positional data. Within this zone, the positions obtained from measures on single photographs are two to three times more precise than those taken from the best maps.

3. *The Sun's Altitude*

The sun's elevation ϕ is the inclination to the lunar horizon of the direction to the sun. At any instant it is a function of position on the lunar surface.

The sun's selenographic coordinates at $0H$ of each day are given in the ephemeris as the colongitude $c_\odot$ and latitude $b_\odot$. For the instant of photography the values can be derived by simple linear interpolation: $b_\odot$ changes very slowly while $c_\odot$ increases at almost constant rate. The colongitude $c_\odot$ is connected to the sun's selenographic longitude $l_\odot$ by

$$l_\odot + c_\odot = 90°. \tag{3}$$

The sun's standard direction-cosines are

$$\left.\begin{aligned} \xi_\odot &= \cos b_\odot \cos c_\odot \\ \eta_\odot &= \sin b_\odot \\ \zeta_\odot &= \cos b_\odot \sin c_\odot \end{aligned}\right\}. \tag{4}$$

By definition, the angle between the lunar radii through the peak and sun is $90° - \phi$; hence, by a well-known result in direction-cosine analysis,

$$\sin\phi = \cos(90° - \phi) = \xi\xi_\odot + \eta\eta_\odot + \zeta\zeta_\odot. \tag{5}$$

4. *The Finite Distance Correction for the Shadow*

Let (l', b') be the topocentric librations, that is, the selenographic longitude and latitude of the telescope at the instant of exposure. The computation of these by the method of Atkinson (1951) is described in *Comm. LPL* No. 10. The derivation of the elements of the orthogonal matrix

$$\begin{pmatrix} a, & b, & c \\ e, & f, & g \\ i, & j, & k \end{pmatrix}$$

from l' and b' is given in *Comm. LPL* No. 60. The perpendicular distance of the peak from the plane of the limb at any instant is

$$Z = iE + jF + kG \tag{6}$$

in units of the moon's radius. The distance from the plane to the telescope is cosec s', where s' is the augmented semidiameter of the moon. Hence, the distances of the peak and the limb plane from the telescope are cosec $s' - Z$ and cosec s' respectively; clearly the shadow is scaled up in the ratio cosec s': cosec $s' - Z$ in the conical projection on the plane of the limb. Hence the shadow is corrected for this by multiplying its measured length by

$$1 - Z \sin s'.$$

Note an approximation here, since in general the lower end or tip of the shadow is at a distance other than Z from the plane of the limb. However, at the center of face, where Z is large, the variation in Z is small, while when this variation is large, near the limb, Z itself is small. Thus in all cases, the use of the Z of the peak as applying to the entire shadow does not generate errors as large as 1 percent of the shadow.

The finite distance correction is incorporated into the foreshortening correction discussed below.

5. *The Foreshortening of the Shadow*

Let the axis of the telescope pierce the lunar surface at the point A, with coordinates (E_A, F_A, G_A). With sufficient precision, for photographs not covering the entire disk, A can be identified with the lunar surface detail at the center of the field of view. In general its coordinates (E_A, F_A, G_A) are not known and, for practical purposes, are replaced with the values (ξ_A, η_A) interpolated from the grids of the *Orthographic Lunar Atlas* (Kuiper, Arthur, and Whitaker 1961). The set is completed with

$$\zeta_A = +\sqrt{(1 - \xi_A{}^2 - \eta_A{}^2)}. \tag{7}$$

This approximation is more than precise enough for the purpose. The standard direction-cosines of the telescope are

$$\left.\begin{aligned} \xi' &= \cos b' \sin l' \\ \eta' &= \sin b' \\ \zeta' &= \cos b' \cos l' \end{aligned}\right\}, \tag{8}$$

and hence, its rectangular selenodetic coordinates reduced to A as origin are

$$\begin{aligned} &\xi' \operatorname{cosec} s' - \xi_A, \\ &\eta' \operatorname{cosec} s' - \eta_A, \\ &\zeta' \operatorname{cosec} s' - \zeta_A. \end{aligned}$$

Introducting proportionals to these, i.e.,

$$\left.\begin{aligned} x &= \xi' - \xi_A \sin s' \\ y &= \eta' - \eta_A \sin s' \\ z &= \zeta' - \zeta_A \sin s' \end{aligned}\right\}, \tag{9}$$

then the standard direction-cosines of the line from A to the telescope are

$$\left.\begin{aligned} x'' &= x/\sqrt{(x^2 + y^2 + z^2)} \\ y'' &= y/\sqrt{(x^2 + y^2 + z^2)} \\ z'' &= z/\sqrt{(x^2 + y^2 + z^2)} \end{aligned}\right\}. \tag{10}$$

These are therefore the standard direction-cosines of the axis of the telescope in the sense moon-earth. Taking (4) into account, the angle between the sun's rays and the axis of the telescope is found from

$$\cos M_A = \xi''\xi_\odot + \eta''\eta_\odot + \zeta''\zeta_\odot \tag{11}$$

Since the shadows have the same direction as the sun's rays, in projection on the plane of the photograph they are shortened in the ratio $\sin M_A$. This factor is always positive and is conveniently computed from

$$\sin M_A = +\sqrt{(1 - \cos^2 M_A)} \tag{12}$$

without recourse to trigonometric tables or routines. Thus, the measured shadows must be divided by $\sin M_A$ to remove the foreshortening. The true shadow χ, in units of the moon's radius and free of foreshortening and finite distance effects, is

$$\chi = \frac{\Delta x \cdot (1 - Z \sin s')}{r \sin M_A}, \tag{13}$$

where Δx and r are respectively the measured shadow length and the radius of the moon at the scale of the photograph, both in the units of the measures. The derivation of r is given in Section 11.

For a photograph of the entire disk, it is usual and appropriate to assume that the axis is aimed at the center of the disk, in which case M_A is replaced by its selenocentric equivalent M. This is found from

$$\cos M = \xi'\xi_\odot + \eta'\eta_\odot + \zeta'\zeta_\odot. \tag{14}$$

This selenocentric value can differ from the real value M_A by as much as 15′, but a more realistic maximum for the deviation is $\gamma = 10'$. Suppose now that we use M when we should use M_A; what restrictions must we place on M or M_A in order not to introduce an error of more than p percent in the reduced length of the shadow? Clearly, $\sin M$ and $\sin M_A$ must not differ by more than p percent of their values. Hence the condition

$$\sin (M + \gamma) - \sin M \leqslant 0.01\, p \sin M.$$

Taking γ as 10′, this reduces to

$$|\tan M| \geqslant 1/3.44\, p. \tag{15}$$

For 1 percent precision, M is limited to the range

16° to 164°, which is not at all restrictive, since relative height work is never attempted outside this range. However, for a precision of one part in one thousand, M is restricted to the range 70° to 110°. Very little relative height work is attempted outside these limits.

Thus in practice, the error of identifying M_A with M, as Kopal (Kopal *et al.* 1961) has done in using his equation 1–18, has virtually no effect on the results, particularly for the Manchester measures, for which γ is probably much less than 10′. The error of principle should nevertheless be noted.

6. *The Computation of Relative Lunar Altitudes*

In Figure 1, P is the lunar peak and T is the tip of its shadow. The shadow subtends an angle θ at M the center of the moon. Since $M\hat{P}T$ is $90° - \phi$ by definition, then $M\hat{T}P$ is $90° + \phi - \theta$. With all lengths in units of the moon's radius we assume that MT is unity for the present. Then $PT = \chi$ and $MP = 1 + h$, where h is the altitude of P relative to T.

Applying the sine rule to the plane triangle $M\hat{P}T$

$$\left.\begin{aligned} \sin\theta &= \chi\cos\phi \\ h &= \cos(\phi - \theta)\sec\phi - 1 \end{aligned}\right\}. \tag{16}$$

This is the computing scheme used by Schmidt (1878). Its only drawback for desk calculations is the necessity for rather extensive trigonometric tables. To compute to four significant figures, six-place tables are required. For high-speed calculations its requirement for direct and inverse trigonometric routines is something of a disadvantage. These can be avoided without approximation by the use of square root routines, which are about the slowest of the algebraic operations in the computer. The scheme (16) can thus be written as

$$\left.\begin{aligned} \cos\phi &= +\sqrt{(1 - \sin^2\phi)} \\ \sin\theta &= \chi\cos\phi \\ \cos\theta &= +\sqrt{(1 - \sin^2\theta)} \\ \tan\phi &= \sin\phi/\cos\phi \\ h &= \cos\theta + \sin\theta\tan\phi - 1 \end{aligned}\right\}. \tag{17}$$

In practice there is a preference for approximate forms. With these in mind, consider the limiting form of the triangle $M\hat{P}T$ when χ reaches its largest possible value for a given peak. The shadow PT then grazes the surface tangentially at T, and $M\hat{P}T = 90°$. This makes $\phi = \theta$, and from the right-angled triangle $M\hat{P}T$, we have

$$\chi^2_{\max} + 1 = (1 + h)^2$$

or

$$\chi_{\max} = \sqrt{(2h)}. \tag{18}$$

In this situation ϕ has its minimum value and θ its maximum value, and

$$\cos\phi_{\min} = \cos\theta_{\max} = 1/(1 + h),$$

i.e.

$$\sin\phi_{\min} = \sin\theta_{\max} \approx \sqrt{(2h)}. \tag{19}$$

The classical measures (Schmidt 1878) show that no lunar relative altitude exceeds 0.005 of the radius; hence, from (18) no value of χ ever exceeds 0.1. Writing

$$\begin{aligned} \cos\theta &= (1 - \sin^2\theta)^{\frac{1}{2}} \\ &= (1 - \chi^2\cos^2\phi)^{\frac{1}{2}} \\ &= 1 - \tfrac{1}{2}\chi^2\cos^2\phi - \tfrac{1}{8}\chi^4\cos^4\phi - \ldots, \end{aligned}$$

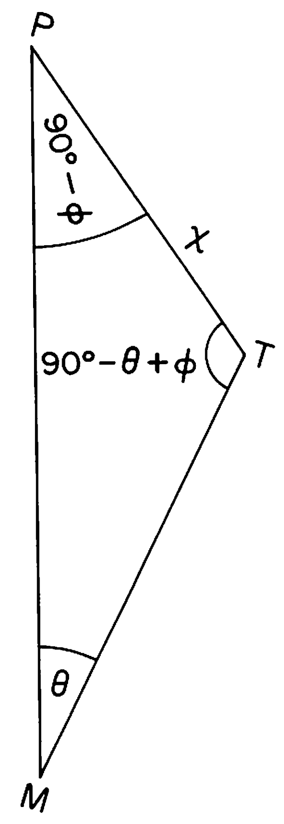

Fig. 1

and substituting this in the expanded form of the second equation in (16), we have

$$h = \chi \sin \phi - \tfrac{1}{2}\chi^2 \cos^2 \phi - \tfrac{1}{8}\chi^4 \cos^4 \phi - \dots \quad (20)$$

For any peak

$$\tfrac{1}{8}\chi^4 \cos^4 \phi < 0.0000125,$$

so the quartic term never exceeds 22 m and only approaches this upper limit for the highest peaks casting their longest possible shadows. In most cases the error will not exceed a few meters and can be neglected. Thus for practical computations we may use the truncated series

$$h = \chi \sin \phi - \tfrac{1}{2}\chi^2 \cos^2 \phi. \quad (21)$$

Since ϕ is derived as its sine, we introduce τ for $\sin \phi$ and write the last as

$$h_m = 1738000(\chi\tau - \tfrac{1}{2}\chi^2 + \tfrac{1}{2}\chi^2\tau^2) \quad (22)$$

for the derivation of the height in meters. The sine is found from (5), i.e.,

$$\tau = \xi\xi_{\odot} + \eta\eta_{\odot} + \zeta\zeta_{\odot},$$

in which ξ, η, ζ are the values for the peak.

The assumption that $MT = 1$ (T lies on the mean sphere) is not generally true. Let the absolute altitude of T be H. Then putting $MT = 1 + H$ and $MP = 1 + H + h$ in Figure 1, we get in place of (16) its rigorous equivalents

$$\sin \theta = \chi \cos \phi/(1 + H) \quad (23)$$

$$h = (1 + H) \cos (\phi - \theta) \sec \phi - H - 1. \quad (24)$$

Consider the error introduced by using (16) in place of (23) and (24). An examination of the latter shows that h is a function of the set ϕ, χ, H through the set ϕ, θ, H. Also in using the approximate equations (16), it is clear that we have introduced an error $\delta H = -H$. The resulting error in h is

$$\delta h = \delta H\left(\frac{\partial h}{\partial \phi}\frac{\partial \phi}{\partial H} + \frac{\partial h}{\partial \theta}\frac{\partial \theta}{\partial H} + \frac{\partial h}{\partial H}\frac{\partial H}{\partial H}\right).$$

In this

$$\delta H = -H,$$

$$\frac{\partial \phi}{\partial H} = 0,$$

$$\frac{\partial h}{\partial \theta} = (1 + H) \sin (\phi - \theta) \sec \phi,$$

$$\frac{\partial h}{\partial H} = \cos (\phi - \theta) \sec \phi - 1 \approx h.$$

so that

$$\partial h = -H\left[h - \frac{\chi \sin (\phi - \theta)}{1 + H}\right].$$

Approximating $(1 + H)$ and $\cos \theta$ to unity and eliminating θ and h with (16) and (21), we are left with

$$\delta h \approx -\tfrac{1}{2}H\chi^2 \cos^2 \phi \quad (25)$$

as the error introduced by the use of the approximation in (16). The distribution of the absolute altitude H is not known, but H cannot much exceed 0.005 of the radius. On the assumption that the shadow never exceeds 0.1, then δh cannot exceed 44 m. This is an extreme case in which H, h, and χ all have their largest values. In practice δh will rarely exceed 10 m.

7. *The Orientation of the Projected Shadow*

The rays of the sun that meet the lunar surface constitute a virtually parallel pencil. Each such pencil viewed in perspective has two vanishing points, and in this case, these are clearly the sun and antisun.

Let M be the angle determined by (14). Then, with sufficient precision for present purposes, M may be taken as the angle at the telescope between the directions to the sun and moon. When $M > 90°$, the sun projects on the plane of the limb of the moon, whereas when $M < 90°$, the antisun is so projected. When $M = 90°$ both vanishing points are projected, but at an infinite distance from the center of the moon's disk. In all three cases the distance (in units of the moon's radius) of the projected vanishing point V from the center of the disk is $\operatorname{cosec} s' |\tan M|$, as shown in Figure 2.

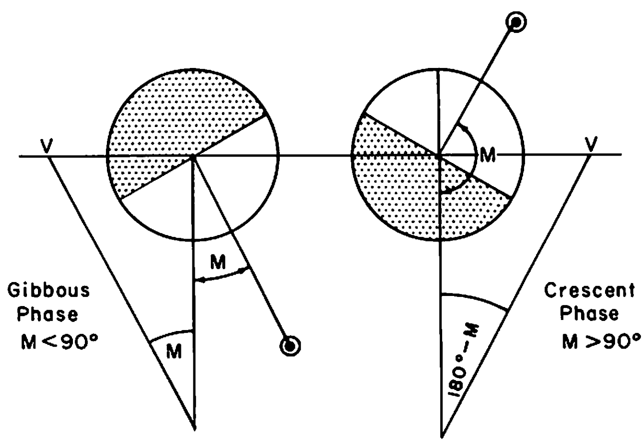

Fig. 2

The projected shadows are, of course, the projections of the sun's rays and are therefore concurrent at whichever point V is projected.

Let $P_1 = (E_1, F_1, G_1)$ and $P_2 = (E_2, F_2, G_2)$ be two well separated points with known positions, and let $A = (E_A, F_A, G_A)$ be the point at which the axis of the telescope meets the lunar surface, as in Section 5. The positions of these are converted to the instantaneous coordinates by

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} a, & b, & c \\ e, & f, & g \\ i, & j, & k \end{pmatrix} \cdot \begin{pmatrix} E \\ F \\ G \end{pmatrix}. \tag{26}$$

and then to the coordinates of their projections on the plane of the limb by

$$\left.\begin{aligned} X' &= X/(1 - Z \sin s') \\ Y' &= Y/(1 - Z \sin s') \end{aligned}\right\}. \tag{27}$$

The set $(\xi_\odot, \eta_\odot, \zeta_\odot)$ for the subsolar point is processed in exactly the same way, treating the direction-cosines as coordinates. At this point we have the four sets (X_1', Y_1'), (X_2', Y_2'), (X_A', Y_A') and $(X_\odot', Y_\odot')$. Let V be the projected vanishing point. It lies in the same direction from the center of face ($X' = 0$, $Y' = 0$) as the point $(X_\odot', Y_\odot')$ if the vanishing point is the sun, and in exactly the opposite direction if the antisun. The discrimination is made automatically by writing

$$S = \frac{\operatorname{cosec} s' \tan M}{\sqrt{(X_\odot'^2 + Y_\odot'^2)}} \tag{28}$$

and

$$\left.\begin{aligned} X_V' &= -SX_\odot' \\ Y_V' &= -SY_\odot' \end{aligned}\right\}. \tag{29}$$

These also place V at the correct distance. The tangent is found without ambiguity and without use of trigonometric routines from

$$\tan M = \frac{+\sqrt{(1 - \cos^2 M)}}{\cos M}. \tag{30}$$

Consider the situation for the photograph of a limited part of the lunar surface with the point A at the center of the field. The direction AV defines the direction of the projected shadows at A, and unless the field is too large, this direction is appropriate for all the shadows on the photograph. Thus for correct orientation, the photograph is oriented so that AV is parallel to measuring motion.

Let the positive (counterclockwise) rotation from the X'-axis to the direction of the line P_1P_2 be α_1. Then

$$\tan \alpha_1 = (Y_2' - Y_1')/(X_2' - X_1'). \tag{31}$$

Similarly if α_2 is the positive rotation from the X'-axis to the line AV,

$$\tan \alpha_2 = (Y_V' - Y_A')/(X_V' - X_A'). \tag{32}$$

The positive rotation from P_1P_2 to AV is

$$\alpha = \alpha_2 - \alpha_1. \tag{33}$$

To commence the measures, the photograph is oriented so that P_1P_2 is parallel to the x-direction. The photograph is then rotated *clockwise* through α to bring AV into the x-direction.

8. *The Measures of the Shadow Lengths and Coordinates*

The first pointing is made on the peak and recorded as (x, y). Without change of y, the second pointing is made on the tip of the shadow and recorded as (x', y). The length of the shadow is

$$\Delta x = |x - x'|. \tag{34}$$

In comparators with digitized output, such as that used at LPL, the failure of the observer to complete the pair (x, y), (x', y) produces a situation in which the subsequent reductions are incorrect. Hence the first computer run scans the output and ensures that the readings are paired, with identical y-readings in each pair. The final outputs of the scanning program are the plate coordinates (x, y) for the peak and Δx for its shadow.

9. *The Reduction of the Coordinate Measures*

Before the shadow measures are commenced, at least four standard points are measured and recorded separately. The (X', Y') values of these are prepared as in (26) and (27). If the original negative is on glass the procedure described in *Comm. LPL* No. 60 may be used to derive the (ξ, η, ζ) for the projection of the peak on the mean sphere.

The situation is different when the original negative is on film. As the film shrinkages may be of the same order or larger than the differential refractions, the procedure of *Comm. LPL* No. 60 is no longer directly applicable. In this case the affine transformation

$$\left.\begin{aligned} X' &= p_1x - q_1y + h \\ Y' &= q_2x + p_2y + k \end{aligned}\right\} \tag{35}$$

is applied to the standard points to derive the six coefficients from two sets of 3 × 3 normals. Equation (35) is then applied to the (x, y) of each peak to derive (X', Y'). With sufficient precision the instantaneous direction cosines are found from

$$\left.\begin{aligned} X &= X' - X'Z' \sin s' \\ Y &= Y' - Y'Z' \sin s' \end{aligned}\right\}. \tag{36}$$

where

$$Z' = +\sqrt{(1 - X'^2 - Y'^2)}.$$

The set (X, Y, Z) is completed with

$$Z = +\sqrt{(1 - X^2 - Y^2)}$$

and converted to (ξ, η, ζ) with the inverse of (26), i.e.,

$$\left.\begin{aligned} \xi &= aX + eY + iZ \\ \eta &= bX + fY + jZ \\ \zeta &= cX + gY + kZ \end{aligned}\right\}.$$

These are the values for the projection of the point on the mean sphere. They may be substituted for the unknown true values of the peak, but with decreasing precision with increasing distance from the center of face.

10. The Selenographic Coordinates of the Tips of the Shadows

In order to pinpoint the two positions involved in the relative height, it is usual to quote ξ and η for both the peak and the tip of its shadow. The values derived in (40) apply to the peak. Let (ξ_T, η_T) be the values for the tip of the shadow. Since the length of PT (in Fig. 1) is χ and its direction-cosines are $(-\xi_\odot, -\eta_\odot, -\zeta_\odot)$, we obviously have

$$\left.\begin{aligned} \xi_T &= \xi - \chi\xi_\odot \\ \eta_T &= \eta - \chi\eta_\odot \end{aligned}\right\}.$$

11. The Radius for the Photograph

The coefficients p_1, q_1 and p_2, q_2 found in the least squares calculation of Section 9, both provide estimates of the radius r of the photograph. These are

$$r_1 = 1/\sqrt{(p_1{}^2 + q_1{}^2)},$$

and

$$r_2 = 1/\sqrt{(p_2{}^2 + q_2{}^2)}.$$

Unfortunately, these are usually slightly discordant because of the affine nature of (35). However since r is required merely for the reductions of the shadow lengths, and since these are parallel to the x-direction, the appropriate value is clearly r_1.

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