NO. 194 ECCENTRICITY AND INCLINATION OF MIRANDA'S ORBIT

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ABSTRACT

Careful re-measurement of all available plates showing Uranus V (Miranda), supplemented by some recently obtained images, shows that this satellite has both a pronounced orbital eccentricity and inclination (to the plane of the other satellites). Observations are sufficient in number and distribution to allow determinations of the precession rates of both pericenter and node, with implications for the dynamical oblateness of Uranus and the gravitational interaction of the satellites. An improved value for the revolution period is a by-product of the investigation. The success of this study is due to the improved precision of the measures resulting from the adoption of a very simple, direct method of measurement.

1. Introduction

Uranus' satellites Miranda, Ariel, and Umbriel display a near-commensurability of mean motions which may be expressed in the form $n_M - 3n_A + 2n_U \simeq 0$. Since an identical relation holds exactly for three Galilean satellites of Jupiter, it is reasonable to suspect that such relations are not due to chance alone, but that they contain information about satellite evolution.

Such consequences motivated an investigation into the remote possibility that the commensurability in the Uranus system appears inexact due to some error in the determination of Miranda's orbital period. We first consulted Van Biesbroeck's (1965) calculations which, apart from two plates taken in the mid-1950's, were based on plates obtained during the 1948-49, 1960, 1961, and 1962 oppositions of Uranus. On checking these figures it soon became apparent that several errors and inconsistencies were present; it was then recalled that at that time, the calculator used for the reductions had a mechanical fault which was not detected until a year or two later. Since the addition of one revolution in the years between these epochs would lower the orbital period from the accepted value of 1.4135 days to the commensurate value 1.4130 days, we decided to repeat Van Biesbroeck's investigation, including a quick check of the position angles of Miranda relative to Uranus. Sections 4 *et seq.* describe how this led to the detection and evaluation of orbital parameters which were previously unresolved.

2. Observations

Following the discovery of Miranda in 1948 by Kuiper (1949), an intensive program of observations was pursued for about one year by Harris (1949) and Kuiper, who obtained about 70 measurable plates with the 82-inch reflector, McDonald Observatory, Texas. Using the same telescope, a few further plates were secured in 1954 and 1955 by Kuiper, and larger numbers in 1960 and 1961 by Kuiper and Whitaker, and in 1962 and 1964 by Van Biesbroeck and Whitaker. Van Biesbroeck also obtained a series of plates with the 61-inch reflector, Catalina Observatory during the 1966 opposition of Uranus, at which epoch the satellite orbits were presented edge-on. Unfortunately, none of these plates contained an image of Miranda.

More recently, Sinton (1972) published a photograph of Miranda taken with the 88-inch reflector, University of Hawaii, using only the light of a methane absorption band. During the 1973 opposition, Whitaker obtained images of Miranda with the 61-inch telescope while testing a special camera which is based on an adaptation of the coronagraph principle. The image of the planetary disk is first occulted by a circular opaque stop; a lens at this location produces an intermediate image of the mirror system, at which point an appropriately-shaped mask occults all planetary light diffracted by the mirror edges, retaining clips and lugs, and the secondary mirror support vanes. Another lens re-images the sky. Figure 1 illustrates a typical image obtained with this camera; because of mediocre seeing, some light from Uranus by-passed the planet stop.

3. Previous Analyses

The 1948-49 group of plates was first measured by Harris (1949) using a standard astrometric measuring machine. He employed normal astrometric reduction techniques, utilizing comparison stars and the other satellites for scale, orientation, and position. The images of Uranus were not measured because of their large size. He obtained improved elements for the four major satellites, and obtained results for Miranda which may be expressed as follows:

P	(orbital period)	1.413487 ± .000007 days
U_O	(mean longitude at epoch)	$105\overset{\circ}{.}8 \pm 0\overset{\circ}{.}8$
e	(eccentricity)	<0.01 (not detected)

where the epoch is 1950.0 (J.D. 2433282.0), and longitude is measured from the ascending node of Miranda's orbit on Uranus' orbit.

Van Biesbroeck (1965) measured 19 plates selected from the 1948-1962 period, giving estimates of the position angle of Miranda relative to Uranus. From these he obtained a value of 1.41347 days for the period, but this result is not valid because of errors introduced by the faulty calculator as already noted.

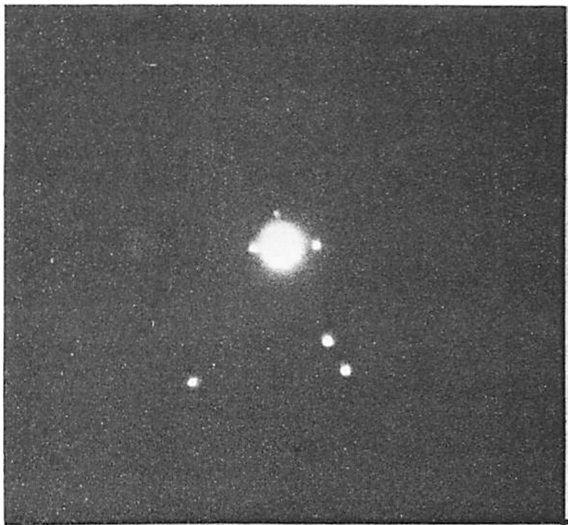
Van Biesbroeck (1970) re-measured a comprehensive selection of plates taken from 1948 through 1966, once again using standard astrometric techniques, and gave results in the form of differential coordinates from Titania (in arc-sec). Astrometric coordinates (α and δ) for Titania were also given. On the basis of these and other measures, Dunham (1971) published the results of an exhaustive analysis of the motions of Uranus' satellites. His results for Miranda may be expressed as follows:

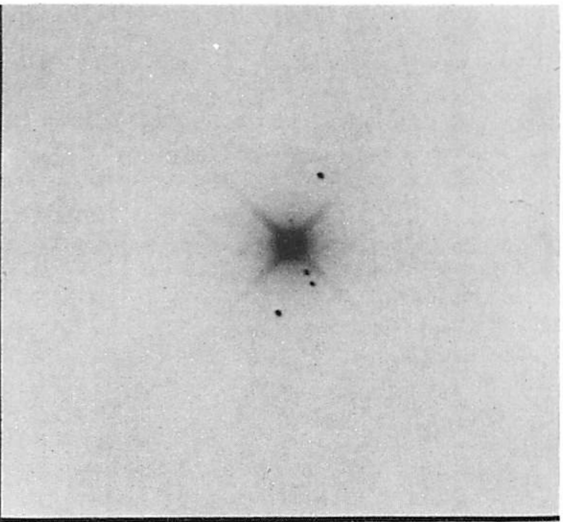
P	1.4134840 ± .0000003 days
U_O	$105\overset{\circ}{.}0 \pm 0\overset{\circ}{.}3$
e	< 0.01 (not detected)

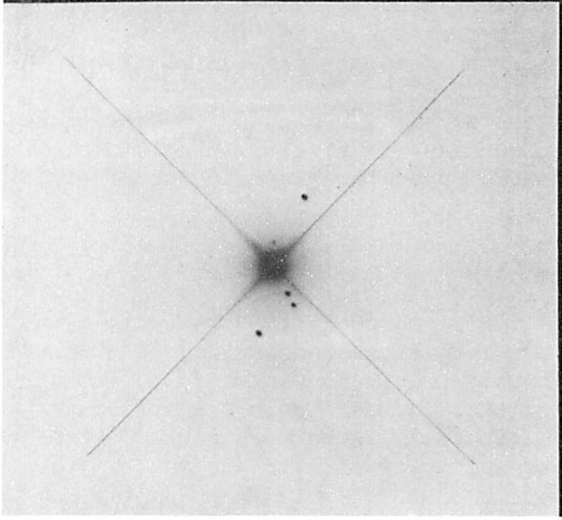
with no other elements detected.

4. Method of Re-measurement

In order to make a quick check of the position angles of Miranda on the 19 plates tabulated by Van Biesbroeck (1965), Whitaker employed the simple expedient of a surplus glass reticle, originally intended for use in a finder telescope, which was marked 0°-360° by 5° increments. Each plate (Fig. 2 shows a typical good-quality image) was first carefully positioned over a pair of thin, orthogonal black lines drawn on tracing plastic placed on a light box. The diffraction cross on each image of Uranus, which is oriented at exactly 45° to true North for the 82-inch reflector, permitted remarkably accurate and reproducible orientation and positioning of each plate (Fig. 3). The reticle was placed symmetrically over the Uranus image-plus-lines and rotated until one of its four cardinal lines bisected the image of Miranda. The position angle was then read directly to the nearest degree by visual interpolation (Fig. 4). Because of the thickness of the diffraction bars, and the fuzzy appearance of many Miranda images, a 3x achromatic lens was found to be optimum for viewing. Measures were repeated with the plates held at various orientations to eliminate possible bias due to personal error. After a little practice, measures for good images were repeatable to ±1° probable error or better, and those of fuzzy images to ± 2° p.e.







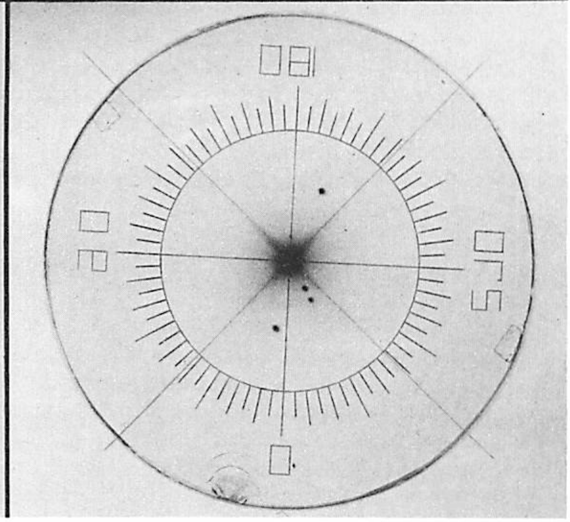


Figure 1 Uranus satellite system as photographed by special camera on the 61-inch reflector, 1973 June 9d 04h 54m U.T. South is up. Lower image at left is a star.

Figure 2 Typical good-quality image ("1" in Table I) of Uranus and satellites from collection of 82-inch plates (CC 544, 1961 Apr 5d 04h 46m U.T.). South is up. See Figure 4 for scale.

Figure 3 Same plate positioned over orthogonal cross.

Figure 4 Same combination of plate plus cross, with the 1-inch diameter reticle almost in correct position. True p.a. = 178°.

5. Reduction of Measures

The new measures were initially reduced exactly as in Van Biesbroeck's Table 1 (1965), using the American Ephemeris tabulations to obtain the epoch of the nearest greatest southern elongation. This gave noticeably smaller residuals, and confirmed Harris' value for the period. However, we realized that the reduction was basically incorrect, since the greatest southern elongation as defined in the American Ephemeris is not a fixed point on the orbit. Furthermore, greatest southern elongation and position angle 180° are not the same point, as stated by Van Biesbroeck (1965, p. 7). Although we had shown to our satisfaction that the period of Miranda was not exactly commensurate with those of Ariel and Umbriel, we decided to investigate matters a little further to see whether the 12.3 year circulation period for contemporaneous conjunctions of these three satellites might not cause some periodic variations in Miranda's period.

6. Further Measurements

In order to obtain better accuracy, we decided to measure all available plates, using the quick and simple method already described. On nights when a large number of images were obtained, one or two of the poorer images were omitted. For the six images obtained in 1972-73, the zero of position angle was determined by calculation of the positions of Titania and Oberon from the American Ephemeris tabulations. Ariel and Umbriel were not used because of systematic divergences from the tabulated positions in each case. Table I lists all plates and images used in the final analysis; the criterion used for retention or rejection is explained in the next Section.

7. Further Reductions

Once again, the general method used by Van Biesbroeck (1965) was employed, except that the datum longitude was taken as Miranda's ascending node on Uranus' orbital plane. The time of next arrival at this longitude was computed from the position angle of each observation, assuming circular motion in Uranus' equatorial* plane and an orbital period of 1.41349 days. These results were compared with the arrival times calculated as integer multiples of 1.41349 days.

It was at once obvious from the run of the residuals that the chosen period was too long. The residuals for 1972-73 observations, compared with the earlier observations, indicated a period nearer 1.41348 days. Furthermore, on plotting the residuals against position in orbit (Figs. 5 and 6), it was clear that systematic trends were present. In particular, the points defined a convincing sinusoid, although the scatter about this curve was still larger than could be accounted for by measuring errors alone. However, when only residuals from 1948-49 were plotted (Fig. 5), the scatter was significantly reduced. We interpreted this sinusoid as representing Miranda's orbital eccentricity for the following reasons: during the 1948-49 epoch the orbit was viewed nearly pole-on so that the residuals would be relatively insensitive to any error in the plane of the orbit, whereas

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* Assumed throughout this paper to be coplanar with the orbits of Ariel and Umbriel.

TABLE I

Measures and Residuals

Plate or Image	Time U.T.	Image Quality	Observed Position Angle°	Residual O-C p.a.°
McD, 82"				
1	1948 Feb 15, 2^h55^m	4	327	-0.9
9	" Mar 1, 2 38	2	108	-2.2
10	" Mar 1, 2 46	1	107	-1.9
11	" Mar 24, 2 00	2	21	+0.4
12	" Mar 24, 2 08	1	18	-1.6
21	" Mar 24, 3 32	3	3	-0.3
22	" Mar 24, 3 46	2	0	-1.2
31	" Mar 25, 2 30	3	118	-0.2
32	" Mar 25, 2 37	3	117	-0.2
33	" Mar 25, 2 42	3	115	-1.2
34	" Mar 25, 2 47	1	114	-1.3
43	" Oct 19, 9 58	3	348	+2.2
44	" Oct 19,10 03	3	345	+0.2
45	" Oct 19,10 09	3	345	+0.2
46	" Oct 21,10 50	3	185	+0.8
47	" Oct 21,10 55	3	183	-0.2
48	" Oct 24, 8 25	2	163	-2.8
49	" Oct 24,11 31	4	132	-1.4
50	" Oct 24,11 45	4	133	+0.5
52	" Oct 24,12 00	4	130	+0.7
53	" Oct 25, 8 30	2	273	-0.1
54	" Oct 25, 8 50	1	271	+1.0
55	" Oct 25, 8 56	2	268	-0.7
56	" Oct 25, 9 06	3	267	+0.5
57	" Oct 25, 9 12	3	265	-0.5
58	" Oct 25,10 08	2	256	+0.6
59	" Oct 25,10 14	2	255	+0.7
60	" Oct 25,10 24	1	253	+0.8
61	" Oct 25,10 30	3	252	+0.9
62	" Oct 25,11 16	1	243	+1.0
63	" Oct 25,11 22	3	242	+1.1
73	" Oct 26,10 06	2	364	+2.0
74	" Oct 26,10 12	2	363	+2.0
77a	" Oct 26,11 30	3	347	-0.2
78	" Oct 26,11 40	1	345	-0.2
79	" Oct 26,11 48	2	344	-0.1
80	" Oct 26,11 55	1	342	-0.9
91b	" Oct 27,11 16	2	93	-1.3
92a	" Oct 27,11 22	2	92	-1.4
93a	" Oct 27,11 42	2	92	+1.6
94a	" Oct 31, 9 44	2	170	+0.8
95a	" Oct 31,10 18	2	164	+1.0
95c	" Oct 31,10 34	2	162	+2.0
96a	" Oct 31,11 01	2	157	+1.2
97a	" Oct 31,11 25	3	151	-0.7
98a	" Nov 6, 8 08	2	98	-2.3
99b	" Nov 6, 8 59	2	90	-1.0
100b	" Nov 6, 9 49	2	82	-0.8
101a	" Nov 6,10 34	3	73	-1.3
102a	" Nov 6,10 51	4	72	+0.7
102b	" Nov 6,10 59	4	72	+1.7
103a	" Nov 6,11 07	4	70	+0.7
103b	" Nov 6,11 15	4	68	+1.0
104a	" Nov 7, 8 34	2	197	0
104b	" Nov 7, 8 41	2	196	0
104c	" Nov 7, 8 48	2	195	0
107b	" Nov 8,10 03	3	292	-0.2
108a	" Nov 8,10 10	3	288	-0.8
108b	" Nov 8,10 17	3	288	+0.2
109a	" Nov 10, 7 36	3	163	-0.6
111b	" Nov 10, 8 11	3	158	+0.6
119a	" Nov 11, 8 44	3	261	+1.1
120	1949 Feb 24, 1 53	4	228	-1.7
122	" Feb 24, 2,09	4	226	-1.6
124	" Feb 24, 3 51	4	208	+0.3
125	" Feb 24, 4 00	2	207	+0.3
126	" Feb 24, 4 07	2	207	+1.4
145a	" Feb 27, 1 51	3	184	-1.2
146b	" Feb 27, 2 32	2	176	-1.8
146d	" Feb 27, 2 48	2	174	-0.7
147a	" Feb 27, 3 47	3	165	+0.9
147c	" Feb 27, 4 00	2	162	-0.1
McD, 82"				
238	1954 Jan 29, 7 00	3	15	+2.0
274	1955 Jan 28, 7 40	2	182	-1.2
276	" Jan 28, 7 45	2	182	-0.4
464	1960 Apr 15, 3 24	4	215	+1.5
465	" Apr 15, 3 28	4	213	+0.2
467	" Apr 15, 3 40	4	213	+1.5
468	" Apr 15, 4 22	1	208	+1.0
469	" Apr 15, 4,28	2	206	-0.3
470	" Apr 15, 4 56	1	204	0
471	" Apr 15, 5 02	1	203	-0.5
472	" Apr 15, 6 11	2	196	-0.9
474	" Apr 16, 2 19	1	358	-1.4
475	" Apr 16, 2 24	1	358	-0.8
476	" Apr 16, 2 33	1	357	-1.2
479	" Apr 16, 2 55	1	356	+0.5
480	" Apr 16, 3 09	1	355	+0.8
481	" Apr 16, 3 14	1	354	+0.6
482	" Apr 16, 4 13	3	347	+1.7
539	1961 Apr 5, 3 42	3	183	-1.2
540	" Apr 5, 3 47	3	182	-1.7
541	" Apr 5, 3 52	3	182	-1.4
542	" Apr 5, 3 57	3	182	-1.2
543	" Apr 5, 4 37	1	178	-1.3
544	" Apr 5, 4 46	1	178	-0.2
546	" Apr 7, 5 24	2	9	-1.6
547	" Apr 7, 5 31	1	9	-1.1
549	" Apr 7, 5 45	3	8	-0.7
550	1962 Mar 27, 6 37	1	193	+0.3
551	" Mar 27, 6 42	1	192	-0.3
552	" Mar 27, 6,47	1	192	+0.1
553	" Mar 27, 6 51	1	192	+0.5
554	" Mar 27, 7 14	2	191	+0.7
555	" Mar 27, 7 18	1	190	+0.1
556	" Mar 27, 7 21	2	190	+0.5
557	" Mar 27, 7 45	2	188	-0.2
558	" Mar 27, 7 50	1	188	+0.2
559	" Mar 27, 7 53	2	187	-0.4
573	" Mar 29, 7 03	3	25	+0.2
574	" Mar 29, 7 07	2	24	-0.3
575	" Mar 29, 7 11	2	24	+0.2
576	" Mar 29, 7 15	3	24	+0.5
577	" Mar 29, 7 18	2	23	-0.4
591	" Apr 25, 3 42	3	25	+0.3
592	" Apr 25, 3 47	1	25	+0.7
593	" Apr 25, 3 56	1	25	+1.2
594	" Apr 25, 4 01	1	24	+0.8
664a	1964 May 30, 3 08	2	34	-0.2
665a	" May 30, 3 16	2	34	+0.5
666a	" May 30, 3 22	2	34	+1.1
667b	" Jun 2, 3 04	3	18	+0.5
668b	" Jun 2, 3 10	1	18	+0.8
669a	" Jun 2, 3 14	1	17	0
670h	" Jun 2, 3 25	1	17	+0.6
671a	" Jun 2, 3 29	1	17	+0.8
672a	" Jun 2, 3 37	1	17	+1.1
Sinton				
88" H.	1972 Mar 17, 10 37	2	7	+0.2
61" Catalina				
W1	1973 May 8, 7 06	2	345	+0.7
W2	" May 9, 5 14	1	200	0
W3	" May 28, 4 33	3	3	-1.0
W4	" Jun 9, 4 54	1	185	+1.0
W5	" Jun 9, 5 15	1	186	+0.1

McDonald 82-inch, CC series; Catalina 61-inch W(hitaker) series

Image Quality:

1. Sharp, small, or circular
2. Fuzzy, large, or asymmetrical
3. Very faint
4. In or touching diffraction ray or planetary halo

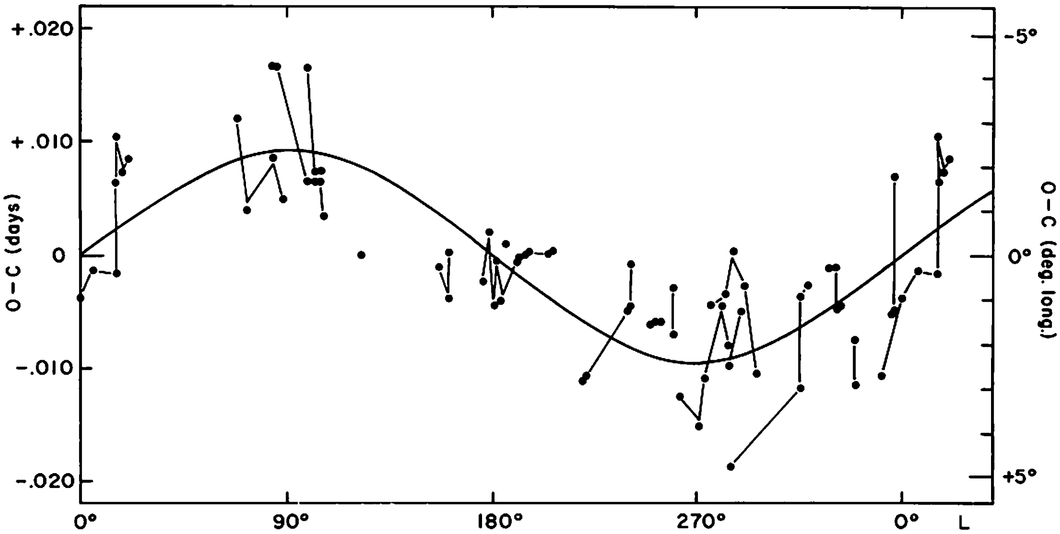

Figure 5 Plot of deviations from circular motion against position in orbit (L, longitude) for 1948-49(approx. pole-on) observations. Results from a single night are joined. The curve represents eccentricity e = 0.02

later residuals would be expected to reflect such orbital inclination because the Earth crossed the Uranus equator in 1966. We estimated the eccentricity to be about 0.02, with the pericenter at 180° longitude.

The scatter in the post-1949 residuals could then be considerably reduced if Miranda's orbit was assumed to be inclined to Uranus' equator by about 7°, with its ascending node 90° from the datum longitude. However, this orbit was rejected for two reasons. First, the residuals were still too large to be accounted for by measuring errors. Second, the oblateness of the planet, with $J_2 \simeq 0.012$ (Dunham 1971), would cause precession of both the apse and the node with periods on the order of 6 years (see Section 8). Thus, a model which neglects precession would not be satisfactory. It was suspected at this point in our investigation that the surprisingly good fit of a fixed inclined orbit was due to the fact that the bulk of the observations were made in three groups at intervals of 12 years, thereby obscuring any 6-year precession.

We next computed the change in each residual as a function of orientation of the orbital plane. With this information we were able to obtain by graphical methods estimates of precession rates and of a plane orientation at epoch which reduced the residuals to acceptable levels. The following list of orbital parameters summarizes our estimate:

P	(orbital period)	1.413480 to 1.413483 days
U_o	(mean longitude at epoch)	107°
e	(eccentricity)	0.02
$\tilde{\omega}_o$	(Longitude of pericenter at epoch)	210°
P_a	(apsidal precession period)	14.6 to 15.1 yr. (direct)
i	(inclination to Uranus' equator)	3.5° to 4.5°
Ω	(longitude of ascending node on Uranus' equator at epoch)	137°
P_n	(nodal precession period)	14.6 to 17.2 yr. (retrograde)

The epoch for U_0 and $\tilde{\omega}_0$ is taken as 1950.0 (J.E.D. 2433282.0); for Ω it is taken as 1962.0, a date which was much closer to edge-on apparition of the orbit and thus allowed a more precise estimate of Ω. Longitudes are measured from the ascending node of Miranda's orbit on Uranus' orbit.

Using the IBM 1130B computer, Greenberg devised and executed a program that adjusted the eight parameters to obtain the least squares of the observed-minus-calculated values of the position angles. The observations were weighted on a 1 or 0 basis: measurements of 133 good images of Miranda were weighted equally, while eight measurements of poor images were disregarded. Images were considered poor if they were pear-shaped due to poor guiding or mirror distortion, if they were of dubious identity or if the positions of the other satellites indicates gross clock errors. The residuals for these poor images were unacceptably large.

Miranda's orbital plane was assumed to precess relative to the equator of the planet. That equator's inclination and node longitude, referred to the Earth's equator and equinox of 1950.0, were taken as $74^{\circ}\!.96$ and $166^{\circ}\!.72$ respectively (Dunham 1971). The direction of Uranus relative to the Earth, at the instant of each observation, was calculated from the orbital elements of the two planets given in the Explanatory Supplement (1961).

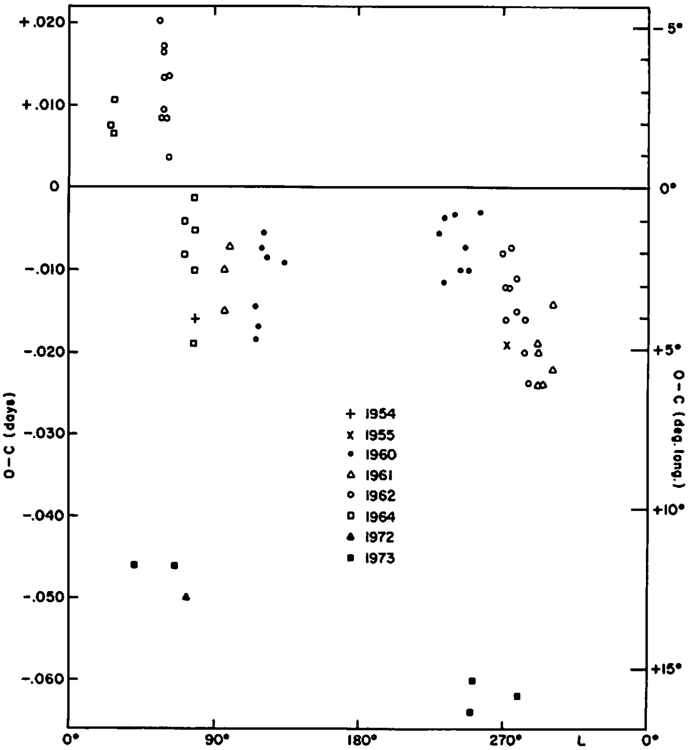

Figure 6 Plot of deviations from circular motion with P = 1.41349 days against position in orbit for 1954-1973 observations

The following parameters (shown with their probable errors) give the best fit:

P	(orbital period)	1.4134823 ± .0000005 days
U_o	(mean longitude at epoch)	106°.48 ± 0°.07
e	(eccentricity)	0.017 ± .001
$\tilde{\omega}_o$	(longitude of pericenter at epoch)	199° ± 2°
P_a	(apsidal precession period)	14.2 ± .2 yr. (direct)
i	(inclination to Uranus' assumed equator)	3°.36 ± 0°.26
Ω	(longitude of ascending node on Uranus' equator at epoch)	136°.4 ± 3°.4
P_n	(nodal precession period)	15.8 ± .5 yr. (retrograde)

with symbols and epochs as given for the graphical solution. Reducing Ω to epoch 1950.0 for uniformity gives Ω_o = 49°.6 ± 10°.6.

With these parameters, the root mean square of the residuals in position angle is ±1°, in agreement with our *a priori* estimate of the measurement precision. Figure 7 gives a plot of these residuals against date.

8. Discussion

A. Measuring technique

The apparent lack of sensitivity of previous measurements to Miranda's eccentricity and inclination can be attributed largely to the relatively high magnification of standard astrometric measuring machines (typically 20x). At that power, many of the images of Miranda appear as scarcely discernible fuzz-balls, and bisection of the Uranus image is impossible. Thus orbital computations have to be made indirectly from the positions of the other satellites. A second factor is the relatively large scale (7.4 arc-sec/mm) at the Cassegrain focus of the 82-inch, which causes a paucity of field stars and hence unreliable scale and orientation determinations.

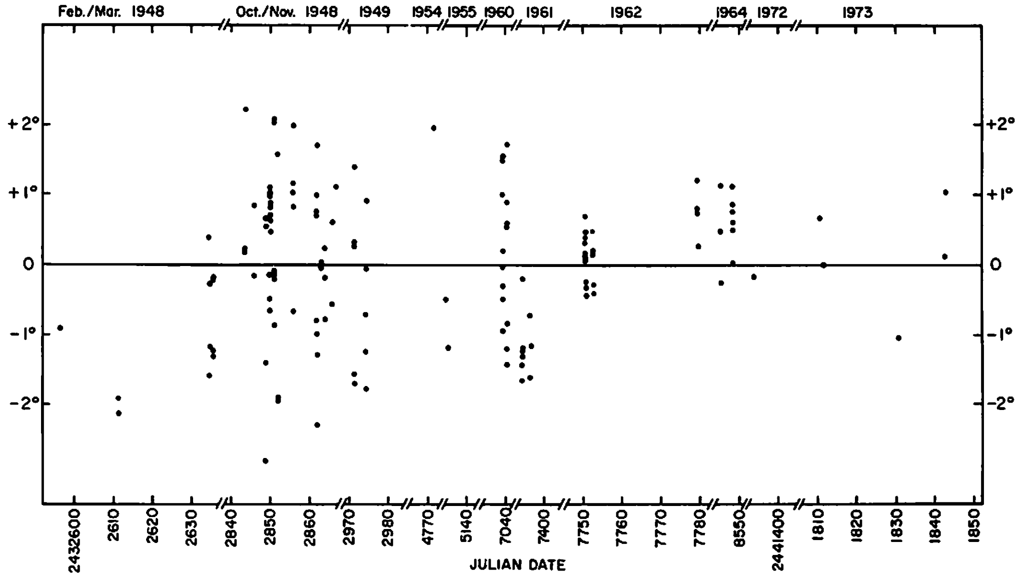

Figure 7 Plot of final O-C residuals in position angle against date

It is apparent that the combined errors arising from these causes are large enough to mask the effects of the eccentricity and tilt of Miranda's orbit, and that the direct measurements made as described in Section 4 are considerably more accurate and reliable.

One possible source of systematic error in our measurement method would be a misalignment of the diffraction cross from its assumed orientation of 45°. Numerical experiments indicate that changing each measurement of Miranda's position angle by 1° (to compensate for an imaginary orientation error for the diffraction cross) would alter the least-square-fit parameters by amounts on the order of their probable errors. The necessity for such a change is unlikely. It would increase the sum of the squares after best fit by 20%. Moreover, Whitaker has determined the orientation of the cross to be 45° ± ~0.1° by comparing it with the shift of star positions on plates with multiple exposures made at constant declination and different right ascensions.

B. Miranda's orbit

For some time, the five satellites of Uranus have enjoyed the reputation of forming the most orderly system known, with eccentricities and inclinations close to or below the limits of detectability. Our study shows that Miranda is an exception to this remarkable symmetry. Moreover, the Uranus system now contradicts the generality that outer satellites of the major planets tend to have the more irregular orbits. In this sense, the discovery of Miranda's inclination and eccentricity is important evidence for any comprehensive theory of the dynamical properties of the solar system.

Miranda's theoretical apsidal precession rate, $\dot{\tilde{\omega}}$, is given (in degrees/yr.) to a reasonable approximation by

$$\dot{\tilde{\omega}} = 4632\, J_2 + 4.65 \times 10^4 m_T$$

where the first term represents secular precession due to the oblateness of the planet (J_2 being the dynamical oblateness coefficient) and where the second term represents secular precession due to the other satellites (m_T being the mass of Titania expressed in units of the mass of Uranus) (Dunham 1971). The nodal precession rate is given by $\Omega = -\dot{\tilde{\omega}}$.

Similar equations for the measured apsidal precession of Ariel and Titania in terms of J_2 and m_T allowed Dunham to solve for these constants, yielding

$$J_2 = 0.012 \pm 0.001 \text{ or } 0.034 \pm 0.02$$

$$m_T = (1.0 \pm 0.7) \times 10^{-4}.$$

Two values for J_2 were given because of an ambiguity in the determination of Ariel's precession rate. However, Dunham accepts the lower value as being the more reasonable. With these values, Miranda's precession rate should be $\tilde{\omega} = 60° \pm 8°$, corresponding to a period of 6 yr ± 1 yr. Our measurements are not consistent with this period. Using the secular theory, the longer periods that we obtain indicate that J_2 should be about half of Dunham's value. On the other hand, the significant difference between our values for the nodal and apsidal precession periods indicates that the secular theory may be inadequate. As a tentative explanation, we would point out that the circulation period for the Miranda-Ariel-Umbriel near-commensurability is about 12.3 yr and that Dunham's period for the Ariel precession is about 20 yr, both of the order of magnitude of Miranda's apparent precession rates. It is thus conceivable that long-period perturbations, disregarded in the secular theory, may be important. Dr. G. P. Kuiper points out to us the possibility that the plane of Uranus' equator may actually be inclined to the mean plane of the four major satellites, a circumstance which could conceivably account for the observed precession rates. An analogous situation exists in the planetary orbits, where the solar equator and the mean orbital plane of Mercury are both inclined about 7° to the invariable plane of the Solar System.

In Section 7 we described our original explanation of the surprisingly good fit of a model with non-zero inclination and no precession. Since the precession periods are not the expected 6 yr periods equal to 1/2 the intervals between observation blocks, we must modify the explanation: apparently, the precession periods are close enough to the observation intervals to explain the fit of the fixed plane model. It should be emphasized that there are sufficient observations to define the precession rates unambiguously.

It is possible, however, that other long-period variations due to the near-commensurability have been disguised by the 12-yr spacing of observations. A few plates showing Miranda at well-distributed points in its orbit, taken at each opposition of Uranus over the next few years, could resolve this problem as well as confirm the orbital parameters. Naturally, in light of our success, we would suggest that such future plates be measured by means of our position-angle method.

Furthermore, our measurement technique might be applied to past and future images of the other Uranus satellites in order to improve, possibly, the determination of their orbits and, more specifically, to search for long-period effects of the near-commensurability.

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