Today:

• Radiative Transfer Equations / Approximate solutions

• GPI project (due in a week!)
Last time …

• Intensity

• Intensity “moments”

• The radiative transfer eq.

\[
\frac{1}{c} \frac{\partial I_\nu}{\partial t} + \mathbf{n} \cdot \nabla I_\nu = \epsilon_\nu - \chi_\nu I_\nu \\
= \chi_\nu (S_\nu - I_\nu)
\]

• source function

\[
S_\nu = \frac{\epsilon_\nu}{\chi_\nu}
\]

• optical depth

\[
\tau_\lambda = \int_0^s \chi_\lambda ds
\]

• radiative equilibrium

\[
\int_0^\infty \chi_\nu S_\nu d\nu = \int_0^\infty \chi_\nu J_\nu d\nu
\]

• diffusion approximation
The basic atmosphere recipe:

1. **Initial $T(r)$ structure**
2. **Hydrostatic Equilibrium Constraint:**
   \[ \frac{dP}{dr} = -\rho g \Rightarrow P(r), \rho(r) \]
3. **Solve LTE EOS**
   \[ \Rightarrow \text{occupation numbers} \]
4. **Get the absorption coefficient**
   \[ \kappa_{\lambda}(r) \]
5. **Solve the RTE (PPRTE or SSRTE)**
   \[ \Rightarrow I(\lambda, r) \]
   Moments: \[ J(\lambda, r), H(\lambda, r), K(\lambda, r) \]
6. **Energy Conserved?**
   \[ \text{Total Flux} = \sigma T_{\text{eff}}^{\lambda} ? \]
   - **No**
   - **Yes**
     - Finished $\Rightarrow T(r), P(r)$
     - Spectrum $\Rightarrow F(\lambda, R)$
useful estimates of temperature structure:
useful estimates of temperature structure:
recall the boundary conditions:

\[ I_\lambda(\tau = 0, \mu \leq 0) = 0 \]

\[ \tau_\lambda = 0 \]

\[ I_\lambda(\tau = \tau_{\text{max}}, \mu \geq 0) = I_{bc}^+ \]

\[ \tau_\lambda = \tau_{\text{max}} \]

\[
\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu
\]

\[ \chi_\lambda = \kappa_\lambda + \sigma_\lambda \]

\[ \tau_\lambda = \int_0^s \chi_\lambda ds \]

\[ S_\nu = \frac{\epsilon_\nu}{\chi_\nu} \]
\[ \mu \frac{\partial I_\nu}{\partial \tau_\nu} = I_\nu - S_\nu \]

linear, 1st-order diff. eq. ... use standard integrating factor: \( e^{-\tau_\nu/\mu} \)

\[ \frac{\partial \left[ I_\nu e^{-\tau_\nu/\mu} \right]}{\partial \tau_\nu} = -\frac{1}{\mu} S_\nu e^{-\tau_\nu/\mu} \]

\[ I(\tau_\nu, \mu, \nu) = \int_{\tau_\nu}^{\infty} S_\nu(t) e^{-(t-\tau_\nu)/\mu} dt / \mu \]

so called “formal or integral” solution
Diffusion Approximation at depth:

\[ S_\lambda = B_\lambda \]

\[ \frac{d^n B_\nu}{d\tau_\nu^n} \approx \frac{B_\nu}{\tau_\nu^n} \]

\[ I_\nu(\tau_\nu) = B_\nu(\tau_\nu) + \mu \left( \frac{dB_\nu}{d\tau_\nu} \right) \]

\[ J_\nu(\tau_\nu) = B_\nu(\tau_\nu) \]

\[ H_\nu(\tau_\nu) = \frac{1}{3} \left( \frac{dB_\nu}{d\tau_\nu} \right) \]

\[ K_\nu(\tau_\nu) = \frac{1}{3} B_\nu(\tau_\nu) \]

assume thermal eq.

Taylor Series +

RT “formal” solution

at large tau, drop all but leading terms in series
boundary conditions at large optical depth:

\[ I_\nu(\tau_\nu) = B_\nu(\tau_\nu) + \mu \left( \frac{dB_\nu}{d\tau_\nu} \right) \]
\[ J_\nu(\tau_\nu) = B_\nu(\tau_\nu) \]
\[ H_\nu(\tau_\nu) = \frac{1}{3} \left( \frac{dB_\nu}{d\tau_\nu} \right) \]
\[ K_\nu(\tau_\nu) = \frac{1}{3} B_\nu(\tau_\nu) \]

\[ F_\nu = \frac{4}{3} \left( \frac{dB_\nu}{d\tau_\nu} \right) \]
\[ = -\frac{4}{3} \left( \frac{1}{\chi_\nu} \frac{dB_\nu}{dT} \right) \left( \frac{dT}{dz} \right) \]
\[ = -D \nabla T \]

Now we have expressions for \( I, J, S, H, F, K \) etc. and \( dT/dz \) at lower boundary. BCs are complete.
Grey Atmosphere:
Depth-dependence of Source function and Temperature

\[ \chi_\nu = \chi = \text{const. in } \nu \]

\[ \mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu \quad \Rightarrow \quad \mu \frac{dI}{d\tau} = I - S \]

\[ I = \int_{0}^{\infty} I_\nu \, d\nu \]

\[ \tau = \int_{0}^{\infty} \tau_\nu \, d\nu \]
For Local Thermodynamic Equilibrium:

assume, locally, that: \( S_\lambda = B_\lambda \) (the Planck Function)

At large depth, this is generally true ...
Grey Atmosphere and Depth-dependence of Source function (and Temperature)

\[ \mu \frac{dI}{d\tau} = I - S \]

Rad. eq.

\[ J(\tau) = S(\tau) = B(T(\tau)) = \frac{\sigma}{\pi} T^4(\tau) \]

1st mom. of RTE

\[ \frac{dH}{d\tau} = J - S = J - J = 0 \quad \rightarrow \quad H = (\sigma/4\pi) T_{\text{eff}}^4 \]

2nd mom. of RTE

\[ \frac{dK}{d\tau} = H \]
Grey Atmosphere and Depth-dependence of Source function (and Temperature)

\[ K(\tau) = H\tau + \text{const.} = \frac{1}{4}F\tau + c \]

Recall:

\[ K \propto B \]

\[ T^4(\tau) = \frac{3}{4}T_{\text{eff}}^4(\tau + q(\tau)) \]

Edd. Aprox. \( q = 2/3 \) (pretty close to full solution)

How good is this approximation?

Depends on the opacities and upper B.C.s
Mean Opacities

• Average opacities are useful in connecting grey to non-grey cases

• It is possible to reduce non-grey to a grey problem..

Recall, these are the moments of the RTE:

\[
\begin{align*}
\mu \frac{dI_I}{dz} &= \chi_\nu(S_\nu - I_\nu) \\
\mu \frac{dI}{dz} &= \chi(S - I) \\
\frac{dH_\nu}{dz} &= \chi_\nu(S_\nu - J_\nu) \\
\frac{dH}{dz} &= \chi(S - J) \\
\frac{dK_\nu}{dz} &= -\chi_\nu H_\nu \\
\frac{dK}{dz} &= -\chi H
\end{align*}
\]
Example: Find a mean opacity so that, when integrated over frequency:

\[
\frac{dK_\nu}{dz} = -\chi_\nu H_\nu \quad \Rightarrow \quad \frac{dK}{dz} = -\bar{\chi}H
\]

This also means that \( K(\bar{\tau}) = H\bar{\tau} + c \) will also be valid in the non-grey case.
Flux weighted mean:

\[ \bar{\chi}_F = \frac{\int_0^\infty \chi_\nu H_\nu \, d\nu}{H} \]

Sadly, we do not know the freq.-dep. flux ahead of time. However, it is still useful for calculating radiation pressure:

\[ \frac{dK}{dz} = -\bar{\chi}_F H \]

However, it is still useful for calculating radiation pressure:

\[ P_{\text{rad}} = \left( \frac{4\pi}{c} \right) K \]

and radiation force:

\[ \frac{dP_{\text{rad}}}{dz} = \frac{1}{\bar{\chi}_F} \frac{dP_{\text{rad}}}{d\tau} = \frac{4\pi}{c\bar{\chi}_F} \int_0^\infty \chi_\nu H_\nu \, d\nu = \frac{4\pi}{c} H = \frac{\sigma}{c} T_{\text{eff}}^4 \]
• Rosseland Mean: construct average so that correct value of freq. integrated flux is recovered.

\[ H = \int_0^\infty H_\nu \, d\nu = - \int_0^\infty \frac{1}{\chi_\nu} \frac{dK_\nu}{dz} \, d\nu \equiv - \frac{1}{\overline{\chi}} \frac{dK}{dz} \]

where

\[ \frac{1}{\overline{\chi}} = \frac{\int_0^\infty \frac{1}{\chi_\nu} \frac{dK_\nu}{dz} \, d\nu}{\int_0^\infty \frac{dK_\nu}{dz} \, d\nu} \]

at large optical depth \( K_\nu \propto B_\nu \)
• **Rosseland Mean**: construct average so that correct value of freq. integrated flux is recovered.

at large optical depth $K_\nu \propto B_\nu$

\[
\frac{1}{\bar{\chi}_R} = \frac{\int_0^\infty \frac{1}{\chi_\nu} \frac{dB_\nu}{dT} \, d\nu}{\int_0^\infty \frac{dB_\nu}{dT} \, d\nu}
\]

same approximation we made for diff. approximation:

\[
H_\nu = -\frac{1}{3} \frac{1}{\chi_\nu} \frac{dB}{dT} \frac{dT}{dz}
\]

\[
H = -\frac{1}{3} \frac{1}{\bar{\chi}_R} \frac{dB}{dT} \frac{dT}{dz}
\]
• **Rosseland Mean**: construct average so that correct value of freq. integrated flux is recovered.

At large optical depth we then have:

\[ T^4(\bar{\tau}_R) = \frac{3}{4} T_{\text{eff}}^4(\bar{\tau}_R + q(\bar{\tau}_R)) \]

...can be great approximation even in non-grey case at depth, but quickly breaks down (flux conservation not guaranteed) near the surface.
Log T (K)

Grey T(τ) structure
AH95
without convection
with pressure broadening
with H₂O/2
with TiO/2
without H₂O

Log P_g (dyn cm⁻²)

Allard et al. 1997
• Rosseland Mean Opacity
Rosseland Mean Opacity

Fig. 10.—Partial Rosseland mean opacity for log $R = -3$ with several opacity sources removed one at a time from the computation. Left: High-temperature sources. The solid line is the total opacity, the dashed line is with H b-f and f-f sources removed; the dotted line removes H$^-$, and the dash-dotted line contains no molecules in the computation. Similarly, in the right panel each of the lines is marked with the source that has been removed. See text for a complete discussion.
Ross. mean and GP evolution

\[
\frac{L_{\text{bol}}(t)}{L_\odot} \propto \left(\frac{1}{t}\right)^\alpha M^\beta \kappa^\gamma
\]

see Stevenson (1991) and Burrows & Liebert (1993)
• **Planck & Absorption Means:** defined to yield correct thermal emission:

\[
\int \kappa_\nu B_\nu \, d\nu = \bar{\kappa}_P \int B_\nu \, d\nu = \frac{\sigma}{\pi} T^4 \bar{\kappa}_P
\]

(tot. E absorbed by medium)

\[
\bar{\kappa}_J J = \int \kappa_\nu J_\nu \, d\nu
\]

These are often assumed to be equal … but!
Major distinction between most brown dwarfs and giant planets -- incident stellar flux.

*Outer boundary condition changes*

\[
I_\nu(\mu, \tau = 0) = I^\text{ext}_\nu(\mu), \mu < 0
\]

\[
I^\text{ext}_\nu(\mu) = J^\text{ext}_\nu
\]

\[
J^\text{ext} \equiv \int_0^\infty J^\text{ext}_\nu d\nu = W B(T_*)
\]

Incident specific intensities from star assume isotropic for plane-par. case (kind of nuts if you think about it too carefully!)

Let $W$ be the dilution factor and $T_{\text{star}}$ be the effective temperature of the host star.
RTE:
\[ \mu \frac{dI_\nu}{dz} = \chi_\nu (S_\nu - I_\nu) \]

rewrite source function as...
\[ S_\nu = \frac{\kappa_\nu}{\chi_\nu} B_\nu + \frac{\sigma_\nu}{\chi_\nu} J_\nu \]

take 1st mom. of RTE
\[ \frac{dH_\nu}{dz} = \chi_\nu (J_\nu - S_\nu) = \kappa_\nu (J_\nu - B_\nu) \]

now int. over frequency
\[ \frac{dH}{dz} = \kappa_B B - \kappa_J J \]
\[ \kappa_B = \frac{\int_0^\infty \kappa_\nu B_\nu \, d\nu}{\int_0^\infty B_\nu \, d\nu} \]
\[ \kappa_J = \frac{\int_0^\infty \kappa_\nu J_\nu \, d\nu}{\int_0^\infty J_\nu \, d\nu} \]
\[
\frac{dH}{dz} = \kappa_B B - \kappa_J J
\]

Recall:
\[
\int_0^\infty \kappa_\nu (J_\nu - B_\nu) \, d\nu = 0 \quad \Rightarrow \quad \kappa_J J - \kappa_B B = 0
\]

\[
H = \text{const} \equiv \frac{\sigma}{4\pi} T_{\text{eff}}^4
\]

\[
\bar{\chi}_F = \frac{\int_0^\infty \chi_\nu H_\nu \, d\nu}{H}
\]

Recall:
\[
\frac{dK}{dz} = -\bar{\chi}_F H
\]

\[
K(\tau_F) = H \tau_F + K(0)
\]
define two “Eddington Factors”:

\[ f_K \equiv K/J \]
\[ f_H \equiv K(0)/J(0) \]

and recall that:

\[ B = \left( \frac{\kappa_J}{\kappa_B} \right) J \]

\[ T^4 = \frac{3}{4} T_{\text{eff}}^4 \frac{\kappa_J}{\kappa_B} \left( \frac{1}{3f_K} \tau_H + \frac{1}{3f_H} \right) + \frac{\kappa_J}{\kappa_B} W T_\ast^4 \]

Outer boundary condition
As first cut: take Eddington’s approx. kJ = kB and the flux mean opacs equal Ross. mean and $f_k = 1/3$ and $f_h = 1/\sqrt{3}$

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 \frac{\kappa_J}{\kappa_B} \left( \frac{1}{3f_K} \tau_H + \frac{1}{3f_H} \right) + \frac{\kappa_J}{\kappa_B} WT_*^4$$

For heavily irradiated planets, the right term dominates, and we can determine a “penetration” depth for the extrinsic radiation:

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 \left( \tau + \frac{1}{\sqrt{3}} \right) + WT_*^4$$

$$\tau_{\text{pen}} \approx W \left( \frac{T_*}{T_{\text{eff}}} \right)^4$$

In the strict gray case the temperature follows the usual behavior, the outer layers are set by the incident flux, the deeper layers by the “intrinsic” flux.
In the case of strong irradiation (esp. with diff lambda_max) the mean opacs depart strongly from each other.

\[ T^4 = \frac{3}{4} T_{\text{eff}}^4 \frac{\kappa_J}{\kappa_B} \left( \frac{1}{3f_K} \tau_H + \frac{1}{3f_H} \right) + \frac{\kappa_J}{\kappa_B} WT_*^4 \]

\[ \kappa_J(T, T_*) \approx \frac{\int \kappa_{\nu}(T) W B_{\nu}(T_*) d\nu}{\int W B_{\nu}(T_*) d\nu} = \frac{\int \kappa_{\nu}(T) B_{\nu}(T_*) d\nu}{\int B_{\nu}(T_*) d\nu} \]

\[ T = \gamma W^{1/4} T_* \]

\[ \gamma \equiv (\kappa_J/\kappa_B)^{1/4} \]

This can spell trouble at the surface!
At deeper layers, we can expect the temperature structure to plateau. Note: radiation becomes increasingly isotropic, which drives the flux-weighted mean to smaller and smaller values. Resulting in little change in tau_H with z …

\[ T_{\text{plateau}} \approx \frac{3}{4} T_{\text{eff}}^4 \bar{\tau}_H \]
Interesting case for strongly irradiated Exoplanets

\[ T = \gamma W^{1/4} T_*, \]

where

\[ \gamma \equiv (\kappa_J / \kappa_B)^{1/4}. \]

\[ \kappa_J(T, T_*) \approx \frac{\int \kappa_\nu(T) W B_\nu(T_*) d\nu}{\int W B_\nu(T_*) d\nu} = \frac{\int \kappa_\nu(T) B_\nu(T_*) d\nu}{\int B_\nu(T_*) d\nu} \]

\[ T / T_0 = \gamma(T). \]

see Hubeny et al. (2003)
Approximate Solutions for Irradiate Atmospheres:

\[ T^4 = \frac{3}{4} T_{\text{eff}}^4 \left( \tau + \frac{2}{3} \right) + \mu_0 T_0^4 \left[ 1 + \frac{3}{2} \left( \frac{\mu_0}{\gamma} \right)^2 \right. \\
\left. - \frac{3}{2} \left( \frac{\mu_0}{\gamma} \right)^3 \ln \left( 1 + \frac{\gamma}{\mu_0} \right) - \frac{3}{4} \frac{\mu_0}{\gamma} e^{-\gamma \tau/\mu_0} \right] \]

Hansen (2008)
Also, Guillot (2010)
Parmentier & Guilot (2014)
Approximate Solutions for Irradiate Atmospheres:

Hansen (2008)  

Barman et al. (2005)
Figure 10. Temperature profiles from our hierarchy of Jupiter models and from observations (solid) (Moses, et al. 2005). Model “a” (dotted) is our purely radiative equilibrium model without solar attenuation (which goes super adiabatic in troposphere), model “b” (dash-dot) is our radiative-convective model without solar attenuation, and model “c” (dashed) is our radiative-convective model with solar attenuation.

\[
\sigma T^4(\tau) = \frac{F_\odot}{2}\left[ 1 + \frac{D}{k_1} + \left( \frac{k_1}{D} - \frac{D}{k_1} \right) e^{-k_1 \tau} \right] + \frac{F_\odot}{2}\left[ 1 + \frac{D}{k_2} + \left( \frac{k_2}{D} - \frac{D}{k_2} \right) e^{-k_2 \tau} \right] + \frac{F_L}{2}(1+D\tau).
\]  

... and similar equations for conv. zone, etc.

Robinson & Catling (2014)
Moving On:

- LTE EOS
- Chemical Equilibrium
- Atomic & Molecular Line Opacities
- Continuos Opacity Sources
- Connecting Opacities and Radiation to atmospheric structure and composition