Lensing
Strong lensing: distortion at large angular scales
Microlensing

microlensing: distortion at small angular scales (θ ≈ 1 mas)

Einstein Ring

If a lens is exactly (or very closely) aligned with a source there are magnified images of the source at (or near) an angle equal to the Einstein radius.
DISCUSSION

LENS-LIKE ACTION OF A STAR BY THE DEVIATION OF LIGHT IN THE GRAVITATIONAL FIELD

Some time ago, R. W. Mandl paid me a visit and asked me to publish the results of a little calculation, which I had made at his request. This note complies with his wish.

The light coming from a star $A$ traverses the gravitational field of another star $B$, whose radius is $R_o$. Let there be an observer at a distance $D$ from $B$ and at a distance $x$, small compared with $D$, from the extended central line $AB$. According to the general theory of relativity, let $a_o$ be the deviation of the light ray passing the star $B$ at a distance $R_o$ from its center.

For the sake of simplicity, let us assume that $AB$ is large, compared with the distance $D$ of the observer from the deviating star $B$. We also neglect the ellipse (geometrical obseration) by the star $B$, which indeed is negligible in all practically important cases. To permit this, $D$ has to be very large compared to the radius $R_o$ of the deviating star.

It follows from the law of deviation that an observer situated exactly on the extension of the central line $AB$ will perceive, instead of a point-like star $A$, a luminous circle of the angular radius $\beta$ around the center of $B$, where

$$\beta = \sqrt{\frac{R_o}{D}}.
$$

It should be noted that this angular diameter $\beta$ does not decrease like $1/D$, but like $1/\sqrt{D}$, as the distance $D$ increases.

Of course, there is no hope of observing this phenomenon directly. First, we shall scarcely ever approach closely enough to such a central line. Second, the angle $\beta$ will defy the resolving power of our instruments. For, $a_o$ being of the order of magnitude of one second of arc, the angle $R_o/D$, under which the deviating star $B$ is seen, is much smaller. Therefore, the light coming from the luminous circle can not be distinguished by an observer as geometrically different from that coming from the star $B$, but simply will manifest itself as increased apparent brightness of $B$.

The same will happen, if the observer is situated at a small distance $x$ from the extended central line $AB$. But then the observer will see $A$ as two point-like light-sources, which are deviated from the true geometrical position of $A$ by the angle $\beta$, approximately.

The apparent brightness of $A$ will be increased by the lens-like action of the gravitational field of $B$ in the ratio $q$. This $q$ will be considerably larger than unity only if $x$ is so small that the observed positions of $A$ and $B$ coincide, within the resolving power of our instruments. Simple geometric considerations lead to the expression

$$q = \frac{1}{x} \cdot \frac{1 + \frac{x^4}{24l^2}}{\sqrt{1 + \frac{x^4}{4l^2}}},
$$

where

$$l = \sqrt{\alpha o D E o}.
$$

December 4, 1936

If we are interested mainly in the case $q \gg 1$, the formula

$$q = \frac{1}{x},
$$

is a sufficient approximation, since $\frac{x^2}{l^2}$ may be neglected.

Even in the most favorable cases the length $l$ is only a few light-seconds, and $x$ must be small compared with this, if an appreciable increase of the apparent brightness of $A$ is to be produced by the lens-like action of $B$.

Therefore, there is no good chance of observing this phenomenon, even if dazzling by the light of the much nearer star $B$ is disregarded. This apparent amplification of $q$ by the lens-like action of the star $B$ is a most curious effect, not so much for its becoming infinite, with $x$ vanishing, but since with increasing distance $D$ of the observer not only does it not decrease, but even increases proportionally to $\sqrt{D}$.

Albert Einstein

Institute for Advanced Study,
Princeton, N. J.
\[ \alpha_{GR} = \frac{4G M_L}{c^2 b} \]
lens equation derived from geometry ... (what is the mapping between beta and theta?)

Gaudi (2012)
Recall that:

\[ \alpha_{GR} = \frac{4GM_L}{c^2 b} \]

\[ R_s = \frac{2GM_L}{c^2} \]

\[ \theta_s = \theta_1 - 2R_s \frac{D_{LS}}{D_L D_S} \frac{1}{\theta_1} \]

(the “lens equation”)

(basic quad. eq.)

\[ y_{\pm} = \pm \frac{1}{2} \left( \sqrt{u^2 + 4} \pm u \right) \]
\[ \theta_s = \theta_1 - 2R_s \frac{D_{LS}}{D_L D_S} \frac{1}{\theta_1} \]

\[ \theta_\pm = \frac{1}{2} \left( \theta_s \pm \sqrt{\theta_s^2 + 4\theta_E^2} \right) \]

(lens equation)

\[ \alpha_{GR} = \frac{4GM_L}{c^2 b} \]

\[ R_s = \frac{2GM_L}{c^2} \]

“angular and linear Einstein Radii”

\[ \theta_E = \left[ \frac{\alpha_{GR} D_{LS}}{D_L D_S} \right]^{1/2} \]

\[ R_E = \theta_E D_L \]
\[ \alpha_{GR} = \frac{4G M_L}{c^2 b} \]

\[ R_s = \frac{2G M_L}{c^2} \]

“angular and linear Einstein Radii”

\[ \theta_E = \left[ \alpha_{GR} \frac{D_{LS}}{D_L D_S} \right]^{1/2} \]

\[ R_E = \theta_E D_L \]

For solar-mass lens halfway to galactic center:

\[ \theta_E \sim 1 \text{ mas} \]

\[ R_E \sim 4 \text{ AU} \]
theta (~ mas) usually too small to resolve spatially, observe total flux

\[ \theta_{\pm} = \frac{1}{2} \left( \theta_s \pm \sqrt{\theta_s^2 + 4\theta_E^2} \right) \]

for \( \theta_s \neq 0 \) -> two images!

see also Paczynski (1996)

image centroid moves in a non-linear path (think astrometry)
Microlensing (magnification)

Lensing conserves surface brightness, so think of lensing as a magnification of the surface. Flux = Area x surface brightness!

*magnification* is the ratio of image area to source area.
Microlensing (magnification)

*magnification* is the ratio of image area to source area

\[
u = \frac{\theta_S}{\theta_E}
\]

\[
y = \frac{\theta}{\theta_E}
\]

\[
A = \frac{I \times w}{A_{\pm}} = \frac{1}{2} \left[ \frac{u^2 + 2}{u \sqrt{u^2 + 4}} \right] \pm 1
\]

(mag is time-dependent change in area)

(approximated by projected area above)

Gaudi (2012)
\[ \theta_{\pm} = \frac{1}{2} \left( \theta_s \pm \sqrt{\theta_s^2 + 4\theta_E^2} \right) \]

\[ A_{\pm} = \left| \frac{y_{\pm}}{u} \frac{dy_{\pm}}{du} \right| = \frac{1}{2} \left[ \frac{u^2 + 2}{u\sqrt{u^2 + 4}} \pm 1 \right] \] (time-dependent)

Animation is in the rest-frame of the lens (foreground star)
from Scott Gaudi: [http://www.astronomy.ohio-state.edu/~gaudi/movies.html](http://www.astronomy.ohio-state.edu/~gaudi/movies.html)
single-lens observables

• time of peak-flux

• time of maximum magnification

• “duration”, Einstein-radius crossing time, $t_E$

Events near the galactic bulge: $t_E$ range $\sim$ days to years

$$t_E \simeq 24.8 \text{ days} \left( \frac{M}{0.5 \, M_\odot} \right)^{1/2} \left( \frac{\pi_{\text{rel}}}{125 \, \mu\text{as}} \right)^{1/2} \left( \frac{\mu_{\text{rel}}}{10.5 \, \text{mas year}^{-1}} \right)^{-1}$$

depends on both lens mass and distance
Microlensing Probability
(lensing optical depth)

\[ P = \frac{\text{Area covered by rings}}{\text{Area of sky}} \]

\[ \tau = \frac{1}{\Omega} \int_0^{D_s} n(D_l)\Omega D_l^2 \pi \theta_E^2 dD_l = \int_0^{D_s} n(D_l) D_l^2 \pi \theta_E^2 dD_l. \]

(For lensing < or = theta_E or A > ~ 1.34)
Microlensing Probability
(lensing optical depth)

\[ P = \frac{\text{Area covered by rings}}{\text{Area of sky}} \]

\[ \tau = \frac{1}{\Omega} \int_0^{D_s} n(D_i) \Omega D_i^2 \pi \theta_E^2 dD_i = \int_0^{D_s} n(D_i) D_i^2 \pi \theta_E^2 dD_i. \]
Microlensing Probability
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\[ n = \frac{\rho}{M} \]
\[ \pi \theta_E^2 \propto M \]

\[ \tau = \frac{4\pi GD_s^2}{c^2} \int_0^1 \rho(x) x(1 - x) dx. \]

depends on mass density (not mass function) along line-of-sight.
\[\theta_s = \theta_1 - 2R_s \frac{D_{LS}}{D_L D_S} \frac{1}{\theta_1}\]

\[\theta_\pm = \frac{1}{2} \left( \theta_s \pm \sqrt{\theta_s^2 + 4\theta_E^2} \right)\]

(lens equation)

\[\alpha_{GR} = \frac{4GM_L}{c^2 b}\]

\[\theta_E = \left[ \alpha_{GR} \frac{D_{LS}}{D_L D_S} \right]^{1/2}\]

\[R_E = \theta_E D_L\]

\[R_s = \frac{2GM_L}{c^2}\]
Microlensing Event Rate

\[ \Gamma = \frac{2}{\pi} \frac{\tau}{t_E} ; \quad t_E = \frac{\theta_E}{\mu_{rel}} \]

does depend on mass function along line-of-sight
look to the Bulge!

Observe towards the Galactic bulge to increase stellar density
microlensing is rare ...

- lensing optical depth toward Galactic Bulge is $\sim 10^{-6}$

- median timescale of event $\sim 20$ days

- rate for lens event / star $\sim 10^{-5}$ per year!

- OGLE survey monitors $\sim 2.5\times10^8$ sources over 80 square degrees $\sim 1,500$ events / 8 month observing window.
Microlensing + planets
**Binary Lens**: adds three more parameters:

- mass ratio

- projected separation of binary at time of event

- angle between source-lens trajectory and binary axis at time of event
Lens Source

mapping between source and lens planes (beta <----> theta)

solution to lens magnification for multiple lens:

\[ A_j = \left| \frac{1}{\det J} \right|_{\theta=\theta_j} \]

caucustic curves: source positions where \( \det J = 0 \)
critical curves: image positions

Gaudi (2012)
pretty “caustics”
pretty “caustics”

Caustics from http://www.mpa-garching.mpg.de/mpa/
planet (angular sep.)

host

"caustics" (typically 2 to 3 for one planet)

$q = 0.003$
Microlensing

Planetary perturbation example:

Magnified images along the Einstein ring pass near the planet and are magnified again ("the source crosses the planetary caustic")
For solar-mass lens halfway to galactic center:

\[ \theta_E \sim 1 \text{ mas} \]

\[ R_E \sim 4 \text{ AU} \]
effect of changing planet/source track angle:
effect of changing separation:

http://www.astronomy.ohio-state.edu/~gaudi/Movies/lcp_b.gif
What can you learn?

Planetary Perturbation

• Same as for the main event: measure time of maximum magnification, maximum magnification, and duration

• The duration is proportional to $q^{1/2} t_E$. With $t_E$ from the main event, you get $q = M_p / M_s$

• The time and magnitude of the perturbation give the separation and position angle of the planet
What can you learn?

Planet-Star System

• With the mass ratio \((q)\), you need to find some way to get the lens mass

• Finite source effects, and some assumptions about the source distance, give you a mass-distance relation for the lens

• Measure photometry and/or spectroscopy of the lens itself

• Measure the proper motion of the system

• Detect the microlens parallax

• Detect orbital motion of the planets
Powerful statistical probe of exoplanet population:

- very sensitive to planets > snow line
- magnification does not depend on planet mass (sensitive to low mass planets)
- sensitive to long-period planets (and free-floaters)
- probes galactic planet distribution
- detects multi-planet systems.
Microlensing Highlights: OGLE-2005-BLG-390Lb

$M_p = 5.5 \ M_{\text{earth}}$

$a = 2.6 \ \text{AU}$

$M_s = 0.20 \ M_{\text{sun}}$

(with big errors)

=> Super-Earths must be common

Beaulieu et al. 2006, Nature, 439, 437
Microlensing Highlights:
OGLE-2006-BLG-109Lb

A Jupiter-Saturn analog system around a M = 0.5 M_\text{sun} star

Orbital motion of the outer planet was detected

Gaudi et al. 2008, Science, 319, 927
Microlensing Highlights: Unbound Planets?

1.8$^{+1.7}_{-0.8}$ planetary mass objects at >10 AU per star

Sumi et al. 2011, Nature, 473, 349
TRIPLE MICROLENS OGLE-2008-BLG-092L: BINARY STELLAR SYSTEM WITH A CIRCUMPRIMAR Y URANUS-TYPE PLANET

Poleski et al. (2014)
microlensing and WFIRST

WFIRST-AFTA will:

- Detect 2800 planets, with orbits from the habitable zone outward, and masses down to a few times the mass of the Moon.
- Be sensitive to analogs of all the solar system’s planets except Mercury.
- Measure the abundance of free-floating planets in the Galaxy with masses down to the mass of Mars.
Finite Source Effects & Microlensing Parallax Yield Lens System Mass

- **Finite source effects**
  
  Angular Einstein radius \( \theta_E = \theta_* t_E / t_* \)
  
  \( \theta_* = \) source star angular radius
  
  \( D_L \) and \( D_S \) are the lens and source distances

- **Microlensing Parallax**
  
  (Effect of Earth’s orbital motion)
  
  Einstein radius projected to Observer
  
  OR

- **One of above**
  
  Lens brightness & color (AO, HST)
  
  mass-distance relation \( \Rightarrow D_L \)

\[
M_L = \frac{c^2}{4G} \theta_E^2 \frac{D_S D_L}{D_S - D_L}
\]

\[
M_L = \frac{c^2}{4G} \tilde{r}_E \frac{D_S - D_L}{D_S D_L}
\]

\[
M_L = \frac{c^2}{4G} \tilde{r}_E \theta_E
\]
Figure 3. Left: Keck image of OGLE-2005-BLG-169 in $H$ band ($\sim 24'' \times 21''$). Middle: a zoom on the target showing the lens on the upper left and the source. They are separated by $\sim 61$ mas. The extra star on the right was part of the measured blending in the microlensing light curve. Right: zoom on the target showing the flux contours.
**microlensing and WFIRST**

![Graph showing microlensing and WFIRST parameters]

**Table 1**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Units</th>
<th>Bennett2015</th>
<th>Gould2006</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_E$</td>
<td>days</td>
<td>41.8 ± 2.9</td>
<td>43 ± 4</td>
</tr>
<tr>
<td>$\theta_E$</td>
<td>mas</td>
<td>0.965 ± 0.094</td>
<td>1.00 ± 0.22</td>
</tr>
<tr>
<td>$H_S$</td>
<td>...</td>
<td>18.81 ± 0.08</td>
<td>18.83 ± 0.09</td>
</tr>
<tr>
<td>$t_*$</td>
<td>days</td>
<td>0.0202 ± 0.0017</td>
<td>0.019 ± 0.004</td>
</tr>
</tbody>
</table>

Distance, and the relative lens–source proper motion, $\mu_{rel}$, where:

$$t_E = \frac{\theta_E}{\mu_{rel,geo}}, \quad \theta_E^2 = \kappa M_L \pi_{rel},$$

$$\pi_{rel} = A U \left( \frac{1}{D_L} - \frac{1}{D_S} \right)$$

and

$$\kappa = \frac{4G}{c^2 AU} = 8.144 \text{ mas } M_\odot^{-1}.$$ (1)

$M_H = H_L - A_H - \text{DM} = H_L - A_H - 5 \log \frac{D_L}{10 \text{ pc}}$