

Waves Along $\vec{B}_0$ + Whistlers

equation of waves in magnetized plasma

$$m_e n_0 \frac{\partial \vec{u}_e'}{\partial t} = -en_0 \vec{E}' - \frac{e}{c} n_0 \vec{u}_e' \times \vec{B}_0 \quad (1)$$

$$\nabla \times \vec{E}' = -\frac{1}{c} \frac{\partial \vec{B}'}{\partial t} \quad (2)$$

$$\nabla \times \vec{B}' = -\frac{4\pi n_0 e}{c} \vec{u}_e' + \frac{1}{c} \frac{\partial \vec{E}'}{\partial t} \quad (3)$$

Assume plane wave soln

$$-i\omega m_e n_0 \vec{u}_e' = -en_0 \vec{E}' - \frac{e}{c} n_0 \vec{u}_e' \times \vec{B}_0$$

$$i \vec{k} \times \vec{E}' = -\frac{1}{c} i\omega \vec{B}'$$

$$i \vec{k} \times \vec{B}' = -\frac{4\pi n_0 e}{c} \vec{u}_e' - \frac{i\omega}{c} \vec{E}'$$

take $\vec{B}_0 = B_0 \hat{z}$ $\vec{k} = k \hat{z}$	CASE A: parallel-propagating waves only
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Since there is no component of $\vec{E}'$ along $\vec{B}_0$, we have

$$\boxed{\begin{aligned} u_{ez}' &= 0 \\ E_z' &= 0 \end{aligned}}$$

this gives

$$-i\omega m_e n_0 u_{ex}' = -en_0 E_x' - \frac{e}{c} n_0 u_{ey}' B_0 \quad (4)$$

$$-i\omega m_e n_0 u_{ey}' = -en_0 E_y' + \frac{e}{c} n_0 u_{ex}' B_0 \quad (5)$$

$$-ik E_y' = i \frac{\omega}{c} B_x' \quad (6)$$

$$ik E_x' = i \frac{\omega}{c} B_y' \quad (7)$$

$$-ik B_y' = -\frac{4\pi n_0 e}{c} u_{ex}' - i \frac{\omega}{c} E_x' \quad (8)$$

$$ik B_x' = -\frac{4\pi n_0 e}{c} u_{ey}' - i \frac{\omega}{c} E_y' \quad (9)$$

from (6) & (7) we have

$$E_x' = \frac{\omega}{kc} B_y' \quad (10)$$

$$E_y' = -\frac{\omega}{kc} B_x' \quad (11)$$

Substituting this into (4), (5), (8), & (9) gives

$$-i\omega m_e n_0 u_{ex}' = -\frac{en_0}{c} \left(\frac{\omega}{k} B_y' + u_{ey}' B_0 \right) \quad (12)$$

$$-i\omega m_e n_0 u_{ey}' = \frac{en_0}{c} \left(\frac{\omega}{k} B_x' + u_{ex}' B_0 \right) \quad (13)$$

$$-ik B_y' = -\frac{4\pi n_0 e}{c} u_{ex}' - i \frac{\omega^2}{kc^2} B_y' \quad (14)$$

$$ik B_x' = -\frac{4\pi n_0 e}{c} u_{ey}' + i \frac{\omega^2}{kc^2} B_x' \quad (15)$$

Eq. (14) becomes

$$-ik B_y' \left(1 - \frac{\omega^2}{k^2 c^2}\right) = -\frac{4\pi n_0 e}{c} u_{ex}'$$

$$\Rightarrow B_y' = -\frac{4\pi n_0 e}{kc} \frac{1}{1 - \frac{\omega^2}{k^2 c^2}} i u_{ex}' \quad (16)$$

while (15) becomes

$$ik B_x' \left(1 - \frac{\omega^2}{k^2 c^2}\right) = -\frac{4\pi n_0 e}{c} u_{ey}'$$

$$\Rightarrow B_x' = \frac{4\pi n_0 e}{kc} \frac{1}{1 - \frac{\omega^2}{k^2 c^2}} i u_{ey}' \quad (17)$$

Substituting (16) $\rightarrow$ (12) gives

$$-i\omega m_e n_0 u_{ex}' = -\frac{en_0}{c} \left\{ -\frac{4\pi n_0 e}{c} \frac{\omega}{k^2 - \frac{\omega^2}{c^2}} i u_{ex}' + u_{ey}' B_0 \right\}$$

$$\Rightarrow u_{ex}' = -\frac{4\pi n_0 e^2}{m_e} \frac{1}{k^2 c^2 - \omega^2} u_{ex}' - \frac{1}{\omega m_e c} i u_{ey}'$$

$$= -\frac{\omega_e^2}{k^2 c^2 - \omega^2} u_{ex}' - i \frac{\Omega_e}{\omega} u_{ey}'$$

$$\Rightarrow \left(1 + \frac{\omega_e^2}{k^2 c^2 - \omega^2}\right) u_{ex}' + i \frac{\Omega_e}{\omega} u_{ey}' = 0 \quad (18)$$

while, substituting (17) $\rightarrow$ (13) gives

$$-i\omega m_e n_0 u_{ey}' = \frac{en_0}{c} \left(\frac{4\pi n_0 e}{c} \frac{\omega}{k^2 - \frac{\omega^2}{c^2}} i u_{ey}' + u_{ex}' B_0 \right)$$

$$\Rightarrow u'_{ey} = -\frac{4\pi n_0 e^2}{m_e} \frac{1}{k^2 c^2 - \omega^2} u'_{ey} + \frac{e B_0}{m_e c} \frac{1}{\omega} i u'_{ex}$$

$$\Rightarrow -i \frac{\Omega_e}{\omega} u'_{ex} + \left(1 + \frac{\omega_p^2}{k^2 c^2 - \omega^2}\right) u'_{ey} = 0 \quad (19)$$

$$\Rightarrow \begin{pmatrix} 1 + \frac{\omega_p^2}{k^2 c^2 - \omega^2} & i \frac{\Omega_e}{\omega} \\ -i \frac{\Omega_e}{\omega} & 1 + \frac{\omega_p^2}{k^2 c^2 - \omega^2} \end{pmatrix} \begin{pmatrix} u'_{ex} \\ u'_{ey} \end{pmatrix} = 0$$

This has the dispersion relation

$$\left(1 + \frac{\omega_p^2}{k^2 c^2 - \omega^2}\right)^2 - \left(\frac{\Omega_e}{\omega}\right)^2 = 0$$

$$\Rightarrow 1 + \frac{\omega_p^2}{k^2 c^2 - \omega^2} = \pm \frac{\Omega_e}{\omega}$$

$$\Rightarrow 1 - \frac{\omega_p^2}{\omega^2 - k^2 c^2} = \pm \frac{\Omega_e}{\omega}$$

$$\Rightarrow \frac{\omega_p^2}{\omega^2 - k^2 c^2} = 1 \mp \frac{\Omega_e}{\omega}$$

$$\Rightarrow \boxed{\frac{\omega^2 - k^2 c^2}{\omega_p^2} \left(1 \mp \frac{\Omega_e}{\omega}\right) = 1}$$

← Equation 12.96
in Kruks & Russell