SOLUTIONS TO PROBLEM SET #1

#1 a) $$\underset{\sim}{A} \times (\underset{\sim}{B} \times \underset{\sim}{C}) = \underset{\sim}{A} \times (\epsilon_{ijk} B_j C_k)$$

$$= \epsilon_{mni} A_n (\epsilon_{ijk} B_j C_k)$$

$$= \epsilon_{mni} \epsilon_{ijk} A_n B_j C_k$$

$$= -\epsilon_{min} \epsilon_{ijk} A_n B_j C_k$$

$$= \epsilon_{imn} \epsilon_{ijk} A_n B_j C_k$$

$$= \epsilon_{ijk} \epsilon_{imn} A_n B_j C_k$$

$$= (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}) A_n B_j C_k$$

$$= \delta_{jm} \delta_{kn} A_n B_j C_k - \delta_{jn} \delta_{km} A_n B_j C_k$$

$$= A_n B_m C_n - A_n B_n C_m \qquad (*)$$

$$= (A_n C_n) B_m - (A_n B_n) C_m$$

$$\boxed{\underset{\sim}{A} \times (\underset{\sim}{B} \times \underset{\sim}{C}) = (\underset{\sim}{A} \cdot \underset{\sim}{C}) \underset{\sim}{B} - (\underset{\sim}{A} \cdot \underset{\sim}{B}) \underset{\sim}{C}}$$

b) look @ terms on RHS separately and add.

$$\underset{\sim}{A} \times (\nabla \times \underset{\sim}{B}) = A_n \partial_m B_n - A_n \partial_n B_m$$

(using (*) above and $\nabla \to \partial_i$)

$$\underset{\sim}{B} \times (\nabla \times \underset{\sim}{A}) = B_n \partial_m A_n - B_n \partial_n A_m$$

$$(\underset{\sim}{A} \cdot \nabla)\, \underset{\sim}{B} = A_n \partial_n B_m$$

$$(\underset{\sim}{B} \cdot \nabla)\, \underset{\sim}{A} = B_n \partial_n A_m$$

add these four to give

$$A_n \partial_m B_n - \cancel{A_n \partial_n B_m} + B_n \partial_m A_n - \cancel{B_n \partial_n A_m} + \cancel{A_n \partial_n B_m} + \cancel{B_n \partial_n A_m}$$

$$= A_n \partial_m B_n + B_n \partial_m A_n$$

$$= \partial_m (A_n B_n) = \nabla (\underset{\sim}{A} \cdot \underset{\sim}{B})$$

This proves the identity

c) $$\nabla \times \nabla \times \underset{\sim}{A} = \partial_n \partial_m A_n - \partial_n \partial_n A_m$$

$$= \partial_m \partial_n A_n - \partial_n^2 A_m$$

$$\boxed{\nabla \times \nabla \times \underset{\sim}{A} = \nabla (\nabla \cdot \underset{\sim}{A}) - \nabla^2 \underset{\sim}{A}}$$

d) $$\nabla \cdot (\nabla \times \underset{\sim}{A}) = \partial_i \epsilon_{ijk} \partial_j A_k = \epsilon_{ijk} \partial_i \partial_j A_k^{(1)}$$

$$= \epsilon_{ijk} \partial_j \partial_i A_k \text{ (switching order of deriv.)}$$

$$= -\epsilon_{jik} \partial_j \partial_i A_k^{(2)} \text{ (switching indices on } \epsilon \text{)}$$

$\Rightarrow$ only way (1) & (2) can be true is if $\boxed{\nabla \cdot (\nabla \times \underset{\sim}{A}) = 0}$!

#2 $\underset{\sim}{F} = q\underset{\sim}{E} + \frac{q}{c}\underset{\sim}{v} \times \underset{\sim}{B}$

$\underset{\sim}{E} = 0 \qquad \underset{\sim}{B} = (0,0,B) \qquad B = \text{constant}$

$\Rightarrow \quad m\dot{\underset{\sim}{v}} = \frac{q}{c}\underset{\sim}{v} \times B\hat{z}$

$\Rightarrow \quad m\dot{v}_x = \frac{qB}{c} v_y$

$m\dot{v}_y = -\frac{qB}{c} v_x$

$m\dot{v}_z = 0 \quad \Rightarrow \boxed{v_z = v_{z0} = \text{constant}}$

note:

$m\dot{\underset{\sim}{v}} = \frac{q}{c}\underset{\sim}{v} \times \underset{\sim}{B}$

$\Rightarrow \quad \underset{\sim}{v} \cdot (m\dot{\underset{\sim}{v}}) = \frac{q}{c}\underset{\sim}{v} \cdot (\underset{\sim}{v} \times \underset{\sim}{B}) = 0$

$\Rightarrow \quad m\underset{\sim}{v} \cdot \frac{d\underset{\sim}{v}}{dt} = 0 \quad \Rightarrow \quad \frac{d}{dt}\Big(\underbrace{\tfrac{1}{2}m(\underset{\sim}{v} \cdot \underset{\sim}{v})}_{\text{kinetic energy}}\Big) = 0$

$\Rightarrow \quad \boxed{\text{kinetic energy} = \text{constant}}$

define $\boxed{\Omega = \frac{qB}{mc}}$, and we have

$\dot{v}_x = \Omega v_y \qquad (1)$

$\dot{v}_y = -\Omega v_x \qquad (2)$

take d/dt of (1) and subs. into (2)

$$\ddot{v}_x = \Omega \dot{v}_y = \Omega(-\Omega v_x) = -\Omega^2 v_x$$

$$\Rightarrow \quad \ddot{v}_x + \Omega^2 v_x = 0$$

has solution $\quad v_x = A \cos \Omega t + B \sin \Omega t$

@ $t = 0 \quad v_x = v_{x0} \Rightarrow A = v_{x0}$

$$\Rightarrow \quad v_x = v_{x0} \cos \Omega t + B \sin \Omega t$$

insert into (1) to give

$$-\Omega v_{x0} \sin \Omega t + B \Omega \cos \Omega t = \Omega v_y$$

$$\Rightarrow \quad v_y = B \cos \Omega t - v_{x0} \sin \Omega t$$

@ $t = 0 \quad v_y = v_{y0} \Rightarrow B = v_{y0}$

$$\therefore \quad v_x = v_{x0} \cos \Omega t + v_{y0} \sin \Omega t$$

$$v_y = v_{y0} \cos \Omega t - v_{x0} \sin \Omega t$$

define $\phi_0 = \text{initial phase} = \tan^{-1}\left(\frac{v_{y0}}{v_{x0}}\right)$

$$v_{\perp 0} = \text{initial speed normal to } \underset{\sim}{B} = \left(v_{x0}^2 + v_{y0}^2\right)^{1/2}$$

we have

$$V_x = V_{\perp 0} \cos(\Omega t - \phi_0)$$

$$V_y = -V_{\perp 0} \sin(\Omega_0 t - \phi_0)$$

$$V_z = V_{\parallel 0} \qquad\qquad (V_{\parallel 0} = (v^2 - V_{\perp 0}^2)^{1/2})$$

note that $\dot{\underset{\sim}{x}} = \underset{\sim}{v}$

$$\Rightarrow \quad \dot{x} = v_x = v_{\perp 0} \cos(\Omega t - \phi_0)$$

$$\Rightarrow \quad x = \frac{v_{\perp 0}}{\Omega} \sin(\Omega t - \phi_0) + C$$

$$@ \quad t = 0 \quad x = x_0 \quad \Rightarrow \quad C = x_0 + \frac{v_{\perp 0}}{\Omega} \sin\phi_0$$

$$\therefore \quad x = x_0 + \frac{v_{\perp 0}}{\Omega}\left[\sin(\Omega t - \phi_0) + \sin\phi_0\right]$$

$$\boxed{x = x_0 + r_{g0}\left[\sin(\Omega t - \phi_0) + \sin\phi_0\right]}$$

where $\boxed{r_{g0} = \text{Gyroradius} = \frac{v_{\perp 0}}{\Omega}}$

Similarly $\quad \dot{y} = v_y = -V_{\perp 0} \sin(\Omega t - \phi_0)$

$$\Rightarrow \quad y = \frac{V_{\perp 0}}{\Omega} \cos(\Omega t - \phi_0) + D$$

@ $t=0 \quad y = y_0 \Rightarrow D = -\frac{v_{\perp 0}}{\Omega}\cos\phi_0 + y_0$

$$\therefore \quad \boxed{y = y_0 + r_g\left[\cos(\Omega t - \phi_0) - \cos\phi_0\right]}$$

and, lastly

$$\dot{z} = v_z = v_{z0} = v_{\parallel 0} = \text{constant}$$

$$\Rightarrow \boxed{z = z_0 + v_{\parallel 0}\, t}$$

this completes the complete set of solutions for the orbit of the particle.

#3 Start w/ conservation of magnetic moment

$$\frac{W_\perp}{B} = \text{const}$$

$$\Rightarrow \left(\frac{W_\perp}{B}\right)_{\text{release}} = \left(\frac{W_\perp}{B}\right)_{\text{mirror pt.}}$$

$$\Rightarrow \left(\frac{W \sin^2\alpha}{B}\right)_{\text{release}} = \left(\frac{W \sin^2\alpha}{B}\right)_{\text{mirror pt.}}$$

at release $\alpha = \alpha_0$

$B = B(r_0, \lambda_0)$ $\quad \lambda =$ latitude

at mirror pt. $\alpha = \pi/2$

$B = B(r_m, \lambda_m)$

$W =$ constant.

$$\Rightarrow \frac{\sin^2\alpha_0}{B(r_0, \lambda_0)} = \frac{1}{B(r_m, \lambda_m)}$$

$$B(r, \lambda) = \frac{M}{r^3}\left(1 + 3\sin^2\lambda\right)$$

$$\Rightarrow \frac{r_0^3 \sin^2\alpha_0}{1 + 3\sin^2\cancel{\lambda_0}\ 0} = \frac{r_M^3}{1 + 3\sin^2\lambda_m}$$

the field line equation is

$$r = r_0 \cos^2 \lambda$$

$$\Rightarrow \quad r_M = r_0 \cos^2 \lambda_M$$

Thus, we have

$$r_0^3 \sin^2 \alpha_0 = \frac{r_0^3 \cos^6 \lambda_M}{1 + 3 \sin^2 \lambda_M}$$

$$\Rightarrow \quad \sin^2 \alpha_0 = \frac{\cos^6 \lambda_M}{1 + 3 \sin^2 \lambda_M}$$

must solve for λ_M. This could be done on a computer, or by trial and error. But, since the release pitch angle (70°) is not far from 90°, its mirror pt., it will likely not move far from the equator; so, lets use small angle formulas

for $\lambda_M << 1$

$$\cos \lambda_M \simeq 1 - \frac{1}{2} \lambda_M^2$$

$$\sin \lambda_M \simeq \lambda_M$$

correct to second order in λ_M

we have

$$\sin^2\alpha_0 = \frac{1-3\lambda_m^2}{1+3\lambda_m^2}$$ ← using binomial approx.

$$\Rightarrow \quad (1+3\lambda_m^2)\sin^2\alpha_0 = 1 - 3\lambda_m^2$$

$$\Rightarrow \quad \lambda_m^2(3\sin^2\alpha_0 + 3) = 1 - \sin^2\alpha_0$$

$$\Rightarrow \quad \lambda_m \simeq \left[\frac{1}{3}\left(\frac{1-\sin^2\alpha_0}{1+\sin^2\alpha_0}\right)\right]^{1/2}$$

$\alpha_0 = 70°$

leads to $\lambda_m \simeq 0.14$ radians

$$\Rightarrow \quad \boxed{\lambda_m \simeq 8°}$$

note that this is correct to order $\lambda_m^2 \simeq 0.02$ or, about 2%.

#4 (a) 1MeV proton ; $E = 10^6 \, eV$

$$= 10^6 \times (1.6022 \times 10^{-12} \, erg)$$

$$= 1.6022 \times 10^{-6} \, erg$$

non relativistic

$$\therefore \quad v = \left(\frac{2E}{m}\right)^{1/2} = \left[\frac{(2)(1.6022 \times 10^{-6})}{1.6726 \times 10^{-24}}\right]^{1/2} \frac{cm}{s}$$

$$\boxed{v = 1.384 \times 10^9 \, cm/s}$$

(b) $r_g = v/\Omega$; $\Omega = \frac{qB}{mc} = \frac{(4.8032 \times 10^{-10})(5 \times 10^{-4})}{(1.6726 \times 10^{-24})(2.9979 \times 10^{10})} s^{-1}$

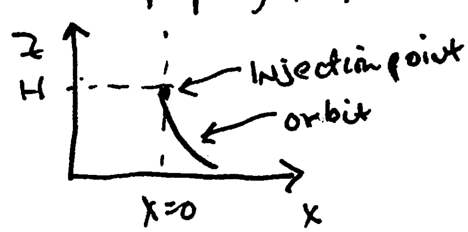
$$= 4.79 \, sec^{-1}$$

$$\therefore \quad r_g = \frac{1.384 \times 10^9}{4.79} cm = 2.89 \times 10^8 \, cm$$

$$\boxed{r_g = 2890 \, km}$$

$$\therefore \quad \boxed{\frac{r_g}{H} = \frac{2890}{50} = 57.8}$$

(c) consider a side view. y is into paper, $\underset{\sim}{B}$ points out of the paper, x is to the right, z is up



Because the gyroradius is so large, the particle only completes a very small part of its orbit before striking the surface. Thus, it is reasonable to estimate that its acceleration in the x direction is constant.

Thus,

$$m \frac{dv_x}{dt} = -\frac{q}{c} v_z B_y = \frac{q}{c} v_z B \simeq \frac{q}{c} v B$$

$$= \text{constant}$$

$$\therefore \quad \frac{dv_x}{dt} = v\Omega = \text{const.}$$

$$\Rightarrow \quad v_x = \cancelto{0}{v_{x0}} + v\Omega t = v\Omega t$$

$$\therefore \quad \frac{dx}{dt} = v_x = v\Omega t$$

$$\Rightarrow \quad x = \cancelto{0}{x_0} + \frac{1}{2} v\Omega t^2 = \frac{1}{2} v \Omega t^2$$

t is just H/v, the time it takes to hit the surface from a height $z = H$. Note that, initially, all its speed is in the downward direction

$$\therefore \quad \Delta x = \frac{1}{2} v \Omega t^2 = \frac{1}{2} v \Omega (H/v)^2 = \frac{1}{2} \frac{H^2 \Omega}{v}$$

$$= \frac{1}{2} \frac{H}{v/\Omega} H = \frac{1}{2} \frac{H}{r_g} H$$

from part (b), we have $H/r_g = \frac{1}{57.8}$, thus

$$\Delta x = \frac{1}{2}\left(\frac{1}{57.8}\right) 50 \text{ km}$$

$$= \frac{50}{2(57.8)} \text{ km}$$

$$\boxed{\Delta x = 0.432 \text{ km}}$$

#5 we note that since B is in $\hat{r}$ direction only and there is no electric field, the equations of motion imply

$$v_r = \text{constant} \qquad (\text{i.e. } m\dot{v}_r = 0 \Rightarrow v_r = \text{const})$$

The equations of motion are written

$$m\dot{\mathbf{v}} = m\ddot{\mathbf{r}} = \frac{q}{c}\frac{K}{r^2}\mathbf{v}\times\hat{r}$$

$$= \frac{q}{c}\frac{K}{r^2}\dot{\mathbf{r}}\times\hat{r}$$

$$\Rightarrow \quad \ddot{\mathbf{r}} = \frac{qK}{mc}\,\frac{\dot{\mathbf{r}}\times\hat{r}}{r^2}$$

$$\ddot{\mathbf{r}} = \frac{qK}{mc}\,\frac{\dot{\mathbf{r}}\times\mathbf{r}}{r^3}$$

Consider the angular momentum vector (divided by m)

$$\mathbf{L} = \mathbf{r}\times\dot{\mathbf{r}} \quad \leftarrow \text{ angular momentum divided by } m$$

note:

$$\frac{d\mathbf{L}}{dt} = \cancelto{0}{\dot{\mathbf{r}}\times\dot{\mathbf{r}}} + \mathbf{r}\times\ddot{\mathbf{r}}$$

$$= \mathbf{r}\times\left[\frac{qK}{mc}\,\frac{\dot{\mathbf{r}}\times\mathbf{r}}{r^3}\right]$$

$$\frac{d\mathbf{L}}{dt} = \frac{qK}{mc}\frac{1}{r^3}\,\mathbf{r}\times(\dot{\mathbf{r}}\times\mathbf{r})$$

vector identity

$$= \frac{qK}{mc}\frac{1}{r^3}\left[(\mathbf{r}\cdot\mathbf{r})\dot{\mathbf{r}} - \underbrace{(\mathbf{r}\cdot\dot{\mathbf{r}})}_{=\,r\dot{r}}\mathbf{r}\right]$$

$$= \frac{qK}{mc}\frac{1}{r^3}\left(r^2\dot{\mathbf{r}} - r\dot{r}\,\mathbf{r}\right)$$

$$= \frac{qK}{mc}\left(\frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}\,\mathbf{r}}{r^2}\right)$$

$$= \frac{qK}{mc}\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)$$

$$\Rightarrow \quad \boxed{\mathbf{L} - \frac{qK}{mc}\frac{\mathbf{r}}{r} = \text{constant}}$$

Thus,

$$\mathbf{r}\times\dot{\mathbf{r}} - \lambda\frac{\mathbf{r}}{r} = \text{const.}$$

$$\lambda = \frac{qK}{mc}$$

or

$$\dot{\mathbf{r}}\times\mathbf{r} + \frac{\lambda\,\mathbf{r}}{r} = \text{const.} = \mathbf{L}'$$

the last term is along $\mathbf{r}$, and the first term normal to $\mathbf{r}$. The angle between $\mathbf{L}'$ & $\mathbf{r}$ is a constant. $\cos\psi = \frac{\mathbf{L}'\cdot\mathbf{r}}{L' r} = \frac{\lambda r}{L' r} = \frac{\lambda}{L'} = \text{const.}$

the path is along the surface of a cone and is a geodesic

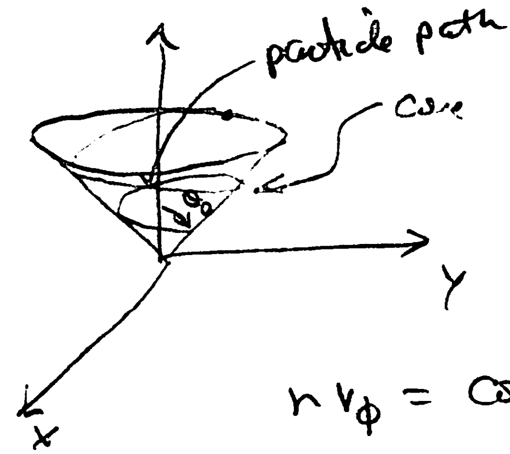
Consider components of $\underset{\sim}{L}'$

$$\dot{\underset{\sim}{r}} \times \underset{\sim}{r} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ v_r & v_\theta & v_\phi \\ r & 0 & 0 \end{vmatrix} = r v_\phi \hat{\theta} - r v_\theta \hat{\phi}$$

$$\therefore \quad \lambda \hat{r} + r v_\phi \hat{\theta} - r v_\theta \hat{\phi} = \text{const.}$$

So, the radial velocity is constant and the other two components of the speed depend inversely on r $\rightarrow$ this is a cone!

take $V_\theta = 0$ as the initial speed. Because $r V_\theta = \text{const.} \Rightarrow V_\theta$ is always zero. $\Rightarrow \theta = \text{const.} = \theta_0$



$$\left.\begin{aligned} r v_\phi &= \text{Const.} \\ v_r &= \text{Const.} \end{aligned}\right\} \text{ path on a cone.}$$