Solutions to problem Set #2

#1 a. Balancing pressure across the tube, we have

$$P_e = P_i + B^2/8\pi$$

external internal

$$\Rightarrow \frac{\rho_e}{\bar{m}} k T_e = \frac{\rho_i}{\bar{m}} k T_i + B^2/8\pi$$

since $T_e = T_i = T$ thermal equilibrium, we have

$$\boxed{\rho_i - \rho_e = -\frac{\bar{m}}{kT}\frac{B^2}{8\pi} < 0}$$

$\Rightarrow$ the flux tube is Buoyant and will rise

b. The MHD momentum eq. in steady state (and static) is

$$-\nabla P + \frac{1}{c}\underset{\sim}{J}\times\underset{\sim}{B} + \rho\underset{\sim}{g} = 0$$

In the direction along the flux tube, we have

$$\underset{\sim}{B}\cdot\left(-\nabla P + \frac{1}{c}\underset{\sim}{J}\times\underset{\sim}{B} + \rho\underset{\sim}{g}\right) = 0$$

$$\Rightarrow \underset{\sim}{B}\cdot(-\nabla P + \rho g) = 0$$

$\Rightarrow$ $\boxed{\nabla P = \rho\underset{\sim}{g} \Rightarrow \text{hydrostatic equilibrium along the tube}}$

c. Since hydrostatic equilibrium holds both inside and outside the flux tube, we have

$$\frac{dP_{e,i}}{dz} = -\rho_{e,i}\, g \qquad (\underset{\sim}{g} = -g\,\hat{z})$$

$$= -\frac{\bar{m} g}{kT} P_{e,i}$$

$$\Rightarrow \quad \frac{d \ln P_{e,i}}{dz} = -\frac{\bar{m} g}{kT}$$

integrating, gives

$$\underbrace{\ln P_{e,i}\Big|_{z_0}^{z}}_{\ln P_{e,i}(z) - \ln P_{e,i}(z_0)} = -\int_{z_0}^{z} dz' \, \frac{\bar{m}\, g(z')}{kT(z')}$$

$$\therefore \quad \ln\left(\frac{P_{e,i}(z)}{P_{e,i}(z_0)}\right) = -\int_{z_0}^{z} dz' \, \frac{\bar{m}\, g(z')}{kT(z')}$$

$$\Rightarrow \quad P_{e,i}(z) = P_{e,i}(z_0)\, e^{-\int_{z_0}^{z} dz' \frac{\bar{m} g(z')}{kT(z')}}$$

Since $P_e = P_i + B^2/8\pi$ everywhere along the tube, we have

$$P_e(z) - P_i(z) = \left[P_e(z_0) - P_i(z_0)\right] e^{-\int_{z_0}^{z} dz' \frac{\bar{m} g(z')}{kT(z')}}$$

$$= \frac{B^2(z_0)}{8\pi}\, e^{-\int_{z_0}^{z} dz' \, \bar{m} g(z')/kT(z')}$$

$$= B^2(z)/8\pi$$

$$\Rightarrow \boxed{B(z) = B(z_0)\, e^{-\frac{1}{2}\int_{z_0}^{z} dz' \frac{\bar{m} g(z')}{kT(z')}}}$$ the desired result

Note that the magnetic field "scale height" is twice the pressure scale height

d. There is pressure balance across the flux tube, but there is also a tension $(B^2/4\pi)$ along the lines of force. The force due to this tension is

$$F_T = A\, B^2/4\pi \qquad (1)$$

the Buoyant force is

$$F_B = A(\rho_e - \rho_i)\, g$$

$$= A \frac{\bar{m} g}{kT}(P_e - P_i)$$

$$= A \frac{\bar{m} g}{kT} \frac{B^2}{8\pi} \qquad (2)$$

In general, we note that

$$\frac{d}{dz}(B^2/8\pi) = d/dz\,(P_e - P_i) = \frac{dP_e}{dz} - \frac{dP_i}{dz}$$

$$= -\rho_e g + \rho_i g$$

$$= -\frac{\bar{m} g}{kT}(P_e - P_i)$$

$$= -\frac{\bar{m}g}{kT}\frac{B^2}{8\pi}$$

$$= -\frac{F_B}{A} \qquad \text{(from eq. 2)}$$

Thus, $F_B = -A\frac{d}{dz}\left(B^2/8\pi\right)$

Recall from (1) that

$$F_T = A\, B^2/4\pi$$

$$\Rightarrow \frac{dF_T}{dz} = \frac{d}{dz} A\frac{B^2}{4\pi} = \frac{d}{dz}(AB)\frac{B}{4\pi}$$

$$= AB\frac{d}{dz}\frac{B}{4\pi}$$

(recall $AB = \Phi$ = Constant by frozen-in condition)

$$= A\frac{d}{dz}\left(\frac{B^2}{8\pi}\right)$$

$$\boxed{\frac{dF_T}{dz} = -F_B}$$ The desired result

#2 The time-independent Vlasov eq. is

$$\underset{\sim}{v} \cdot \nabla f + \frac{\underset{\sim}{F}}{m} \cdot \nabla_v f = 0$$

where $\underset{\sim}{F} = q\underset{\sim}{E} + \frac{q}{c} \underset{\sim}{v} \times \underset{\sim}{B}$

if $f = f(\mathcal{E})$, where $\mathcal{E} = \frac{1}{2} m v^2 + q\phi(\underset{\sim}{x})$

$= \text{constant}$

we have

$$\frac{\partial f}{\partial \underset{\sim}{x}} = \frac{\partial f}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \underset{\sim}{x}} = \frac{\partial f}{\partial \mathcal{E}} \left(q \frac{d\phi(x)}{d\underset{\sim}{x}} \right)$$

$$\frac{\partial f}{\partial \underset{\sim}{v}} = \frac{\partial f}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \underset{\sim}{v}} = \frac{\partial f}{\partial \mathcal{E}} (m\underset{\sim}{v})$$

Thus, we have

$$\underset{\sim}{v} \cdot \nabla f(\mathcal{E}) + \frac{\underset{\sim}{F}}{m} \cdot \nabla_v f(\mathcal{E}) = \underset{\sim}{v} \cdot \left(\frac{\partial f}{\partial \mathcal{E}} q \frac{d\phi}{d\underset{\sim}{x}} \right)$$

$$+ \frac{\underset{\sim}{F}}{m} \cdot \nabla_f \left(\frac{\partial f}{\partial \mathcal{E}} (m\underset{\sim}{v}) \right)$$

$$= q \frac{\partial f}{\partial \mathcal{E}} \underset{\sim}{v} \cdot \nabla \phi + \frac{\partial f}{\partial \mathcal{E}} \underset{\sim}{F} \cdot \underset{\sim}{v}$$

note that $\underset{\sim}{F} = q\underset{\sim}{E} + \frac{q}{c}\underset{\sim}{v}\times\underset{\sim}{B}$

$$\Rightarrow \underset{\sim}{F}\cdot\underset{\sim}{v} = q\underset{\sim}{E}\cdot\underset{\sim}{v}$$

$$\Rightarrow \underset{\sim}{v}\cdot\nabla f(\varepsilon) + \frac{\underset{\sim}{F}}{m}\cdot\nabla_{\underset{\sim}{v}} f(\varepsilon) = \frac{\partial f}{\partial\varepsilon}\left[q\underset{\sim}{v}\cdot\nabla\phi + q\underset{\sim}{E}\cdot\underset{\sim}{v}\right]$$

but $\nabla\phi = -\underset{\sim}{E}$, thus, the last term is $\underline{\underline{0}}$

$$\Rightarrow \underset{\sim}{v}\cdot\nabla f(\varepsilon) + \frac{\underset{\sim}{F}}{m}\cdot\nabla_{\underset{\sim}{v}} f(\varepsilon) = 0$$

$\Rightarrow$ $f(\varepsilon)$ is a solution to time-independent Vlasov eq.

#3 (a) Lorentz force on plasma

$$\underset{\sim}{F}_L = \frac{1}{c}\underset{\sim}{J}\times\underset{\sim}{B}$$

where $\underset{\sim}{B} = \frac{a}{r^2}\hat{r} + \frac{b}{r}\sin\theta\,\hat{\phi}$

$$\Rightarrow \underset{\sim}{J} = \frac{c}{4\pi}\nabla\times\underset{\sim}{B}$$

$$= \frac{c}{4\pi}\left\{\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\phi) - \cancel{\frac{1}{r\sin\theta}\frac{\partial B_\theta}{\partial\phi}}^{\,0}\right)\hat{r}\right.$$

$$+\left(\cancel{\frac{1}{r\sin\theta}\frac{\partial B_r}{\partial\phi}}^{\,0} - \cancel{\frac{1}{r}\frac{\partial}{\partial r}(rB_\phi)}^{\,0}\right)\hat{\theta}$$

$$\left.+\left(\cancel{\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)}^{\,0} - \cancel{\frac{1}{r}\frac{\partial B_r}{\partial\theta}}^{\,0}\right)\hat{\phi}\right\}$$

$$= \frac{c}{4\pi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{b}{r}\sin^2\theta\right)\hat{r}$$

$$= \frac{bc}{2\pi r^2}\cos\theta\,\hat{r}$$

$$\Rightarrow \underset{\sim}{F}_L = \frac{1}{c}\underset{\sim}{J}\times\underset{\sim}{B} = \frac{1}{c}\begin{vmatrix}\hat{r} & \hat{\theta} & \hat{\phi}\\ \frac{bc}{2\pi r^2}\cos\theta & 0 & 0\\ \frac{a}{r^2} & 0 & \frac{b}{r}\sin\theta\end{vmatrix}$$

$$= -\frac{1}{c}\frac{b^2c\cos\theta\sin\theta}{2\pi r^3}\hat{\theta}$$

$$\Rightarrow \boxed{\underset{\sim}{F}_L = -\frac{b^2}{2\pi r^3}\cos\theta\sin\theta\,\hat{\theta}}$$

(b) the electric field in ideal MHD is

$$\underset{\sim}{E} = -\frac{1}{c}\,\underset{\sim}{u} \times \underset{\sim}{B} \quad , \quad \underset{\sim}{u} = V_0 \hat{r}$$

$$\Rightarrow \quad \underset{\sim}{E} = -\frac{1}{c}\begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ V_0 & 0 & 0 \\ \frac{a}{r^2} & 0 & \frac{b}{r}\sin\theta \end{vmatrix} = -\frac{1}{c}\left(-V_0 \frac{b}{r}\sin\theta\,\hat{\theta}\right)$$

$$\Rightarrow \quad \boxed{\underset{\sim}{E} = \frac{V_0}{c}\frac{b}{r}\sin\theta\,\hat{\theta}}$$

the resulting charge distribution from Poisson's law is

$$q^* = \frac{\nabla\cdot\underset{\sim}{E}}{4\pi} = \frac{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,E_\theta\right)}{4\pi}$$

$$= \frac{1}{4\pi r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{V_0}{c}\frac{b}{r}\sin^2\theta\right)$$

$$\boxed{q^* = \frac{V_0\, b\cos\theta}{2\pi r^2 c}}$$

(c) the electric force per unit volume on plasma is

$$\underset{\sim E}{F} = q^*\underset{\sim}{E} = \left(\frac{V_0 b\cos\theta}{2\pi r^2 c}\right)\left(\frac{V_0}{c}\frac{b}{r}\sin\theta\right)\hat{\theta}$$

$$= \left(\frac{V_0}{c}\right)^2 \frac{b^2}{2\pi r^3}\cos\theta\sin\theta\,\hat{\theta}$$

$$= \left(\frac{V_0}{c}\right)^2\left(-\underset{\sim L}{F}\right)$$

; if $V_0 \lll c$, we can neglect this force!

#4 The energy equation is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \frac{1}{\gamma-1}p\right) + \frac{\partial}{\partial x}\left[\left(\frac{1}{2}\rho u^2 + \frac{\gamma}{\gamma-1}p\right)u + \varphi_x\right] = 0$$

$$\Rightarrow \underbrace{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho u^3\right)}_{ⓐ} + \underbrace{\frac{1}{\gamma-1}\frac{\partial p}{\partial t} + \frac{\gamma}{\gamma-1}\frac{\partial}{\partial x}(pu)}_{ⓑ} = -\frac{\partial \varphi_x}{\partial x}$$

look at ⓐ term

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho u^3\right) = \frac{1}{2}\rho u\frac{\partial u}{\partial t} + \frac{1}{2}u\frac{\partial}{\partial t}(\rho u) + \frac{1}{2}\rho u^2\frac{\partial u}{\partial x} + \frac{1}{2}u\frac{\partial}{\partial x}(\rho u^2)$$

recall $\rho\frac{Du}{Dt} = -\nabla p \Rightarrow \rho\frac{\partial u}{\partial t} = -\rho u\frac{\partial u}{\partial x} - \frac{\partial p}{\partial x}$

$$\frac{\partial}{\partial t}(\rho u) = \frac{\partial}{\partial x}(\rho u^2 + p)$$

$$\therefore ⓐ = \cancel{-\frac{1}{2}\rho u^2\frac{\partial u}{\partial x}} - \frac{1}{2}u\frac{\partial p}{\partial x} \cancel{-\frac{1}{2}u\frac{\partial}{\partial x}(\rho u^2)} - \frac{1}{2}u\frac{\partial p}{\partial x} \cancel{+\frac{1}{2}\rho u^2\frac{\partial u}{\partial x}} \cancel{+\frac{1}{2}u\frac{\partial}{\partial x}(\rho u^2)}$$

$$= -u\frac{\partial p}{\partial x}$$

Add this to ⓑ to give

$$-u\frac{\partial p}{\partial x} + \frac{1}{\gamma-1}\frac{\partial p}{\partial t} + \frac{\gamma}{\gamma-1}\frac{\partial}{\partial x}(pu) = -\frac{\partial \varphi_x}{\partial x}$$

$$\Rightarrow \quad -u\frac{\partial p}{\partial x} + \frac{1}{\gamma-1}\frac{\partial p}{\partial t} + \frac{\gamma}{\gamma-1} p \frac{\partial u}{\partial x} + \frac{\gamma}{\gamma-1} u \frac{\partial p}{\partial x} = -\frac{\partial Q_x}{\partial x}$$

$$\Rightarrow \quad \frac{1}{\gamma-1}\frac{\partial p}{\partial t} + \frac{\gamma}{\gamma-1} p \frac{\partial u}{\partial x} + \frac{1}{\gamma-1} u \frac{\partial p}{\partial x} = -\frac{\partial Q_x}{\partial x} \qquad (**)$$

recall $\quad \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}(\rho u) = 0$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x}\right)$$

$$= -\frac{\partial \ln \rho}{\partial t} - u\frac{\partial \ln \rho}{\partial x}$$

Inserting into (**) gives

$$\frac{\partial p}{\partial t} + \gamma p\left(-\frac{\partial \ln \rho}{\partial t} - u\frac{\partial \ln \rho}{\partial x}\right) + u\frac{\partial p}{\partial x} = -(\gamma-1)\frac{\partial Q_x}{\partial x}$$

divide through by p to give

$$\frac{\partial}{\partial t}(\ln p - \gamma \ln \rho) + u\frac{\partial}{\partial x}(\ln p - \gamma \ln \rho) = -\frac{\gamma-1}{p}\frac{\partial Q_x}{\partial x}$$

$$\Rightarrow \quad \frac{D}{Dt}\ln p/\rho^\gamma = -\frac{\gamma-1}{p}\frac{\partial Q_x}{\partial x}$$

$$\Rightarrow \quad \frac{\rho^\gamma}{p}\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = -\frac{\gamma-1}{p}\frac{\partial Q_x}{\partial x}$$

$$\Rightarrow \quad \boxed{\frac{\rho^\gamma}{\gamma-1}\frac{D}{Dt}\left(p/\rho^\gamma\right) = -\frac{\partial Q_x}{\partial x}}$$

the desired result.