Solutions to problem Set #3

1. (a) because the field is force free, $\underset{\sim}{J} \times \underset{\sim}{B} = 0$ and, the problem states gravity is ignored. Thus, the surviving term in the MHD momentum equation is (steady state, no flows)

$$\nabla P - \frac{1}{c}\cancelto{0}{\underset{\sim}{J}\times\underset{\sim}{B}} + \cancelto{0}{\rho \underset{\sim}{g}} = 0 \Rightarrow \nabla P = 0$$

$$\Rightarrow \boxed{P = \text{constant}}$$

(b) $\nabla \times \underset{\sim}{B} = \alpha \underset{\sim}{B}$

$$\Rightarrow \frac{\partial B_z}{\partial y}\hat{x} - \frac{\partial B_z}{\partial x}\hat{y} + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\hat{z} = \alpha \underset{\sim}{B}$$

$$\Rightarrow \frac{dB_z}{dA}\frac{\partial A}{\partial y}\hat{x} - \frac{dB_z}{dA}\frac{\partial A}{\partial x}\hat{y} + \left[\frac{\partial}{\partial x}\left(-\frac{\partial A}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A}{\partial y}\right)\right]\hat{z} = \alpha \underset{\sim}{B}$$

$$\Rightarrow B_x \frac{dB_z}{dA}\hat{x} + B_y\frac{dB_z}{dA}\hat{y} - \nabla^2 A\,\hat{z} = \alpha\underset{\sim}{B}$$

but, from Grad-Shafranov eq. with constant pressure $\nabla^2 A = -\frac{d}{dA}\left(\frac{1}{2}B_z^2\right)$

$$\therefore\ B_x\frac{dB_z}{dA}\hat{x} + B_y\frac{dB_z}{dA}\hat{y} + \underbrace{\frac{d}{dA}\left(\frac{1}{2}B_z^2\right)}_{B_z\, dB_z/dA}\hat{z} = \alpha B$$

$$\Rightarrow \frac{dB_z}{dA}\underset{\sim}{B} = \alpha\underset{\sim}{B}$$

$$\Rightarrow \boxed{B_z = \alpha A + \text{constant}}$$

c) Taking $A = -\cos(kx)\, e^{-qy}$

we have

$$\boxed{\begin{aligned} B_x &= \frac{\partial A}{\partial y} = q\cos(kx)\, e^{-qy} \\ B_y &= -\frac{\partial A}{\partial x} = -k\sin(kx)\, e^{-qy} \\ B_z &= \alpha A \end{aligned}}$$

d) The Grad-Shafranov equation is (constant pressure)

$$\nabla^2 A + \frac{d}{dA}\left(\frac{1}{2}B_z^2\right) = 0$$

$$\Rightarrow \quad \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{d}{dA}\left(\frac{1}{2}B_z^2\right) = 0$$

Upon substitution, we have

$$\frac{\partial^2}{\partial x^2}\left(-\cos(kx)\, e^{-qy}\right) + \frac{\partial^2}{\partial y^2}\left(-\cos(kx)\, e^{-qy}\right) + \frac{d}{dA}\left(\frac{1}{2}\alpha^2 A^2\right) = 0$$

$$\Rightarrow \quad -k^2 A + q^2 A + \alpha^2 A = 0$$

$$\Rightarrow \quad \boxed{q^2 = k^2 - \alpha^2}$$

e)

$$\frac{B^2}{8\pi} = \frac{1}{8\pi}\left(q^2\cos^2(kx)\,e^{-2qy} + k^2\sin^2(kx)\,e^{-2qy} + \alpha^2A^2\right)$$

$$= \frac{1}{8\pi}\left(q^2\cos^2(kx)\,e^{-2qy} + k^2\left(1-\cos^2(kx)\right)e^{-2qy} + \alpha^2A^2\right)$$

$$= \frac{1}{8\pi}\left(q^2A^2 + k^2e^{-2qy} - k^2A^2 + \alpha^2A^2\right)$$

$$\boxed{\frac{B^2}{8\pi} = \frac{k^2}{8\pi}e^{-2qy}}$$

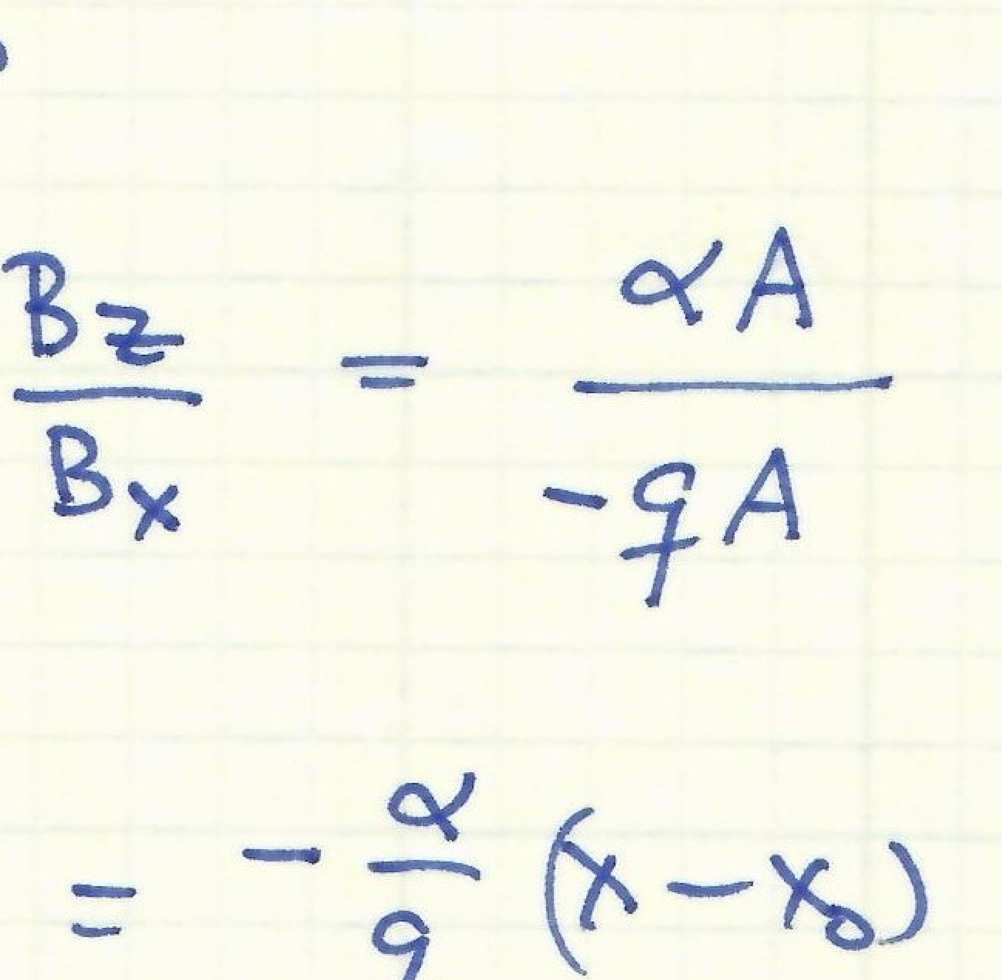
note: from d) $q^2 = k^2 - \alpha^2$

Note: Since $q = (k^2 - \alpha^2)$, if $\underline{\alpha \text{ is increased}}$ and $k$ is fixed, $q$ decreases $\Rightarrow$ $\underline{B^2/8\pi \text{ increases}}$

f) The equations for the field lines in the x-z plane is

$$\frac{dz}{dx} = \frac{B_z}{B_x} = \frac{\alpha A}{-qA} = \frac{-\alpha}{q}$$

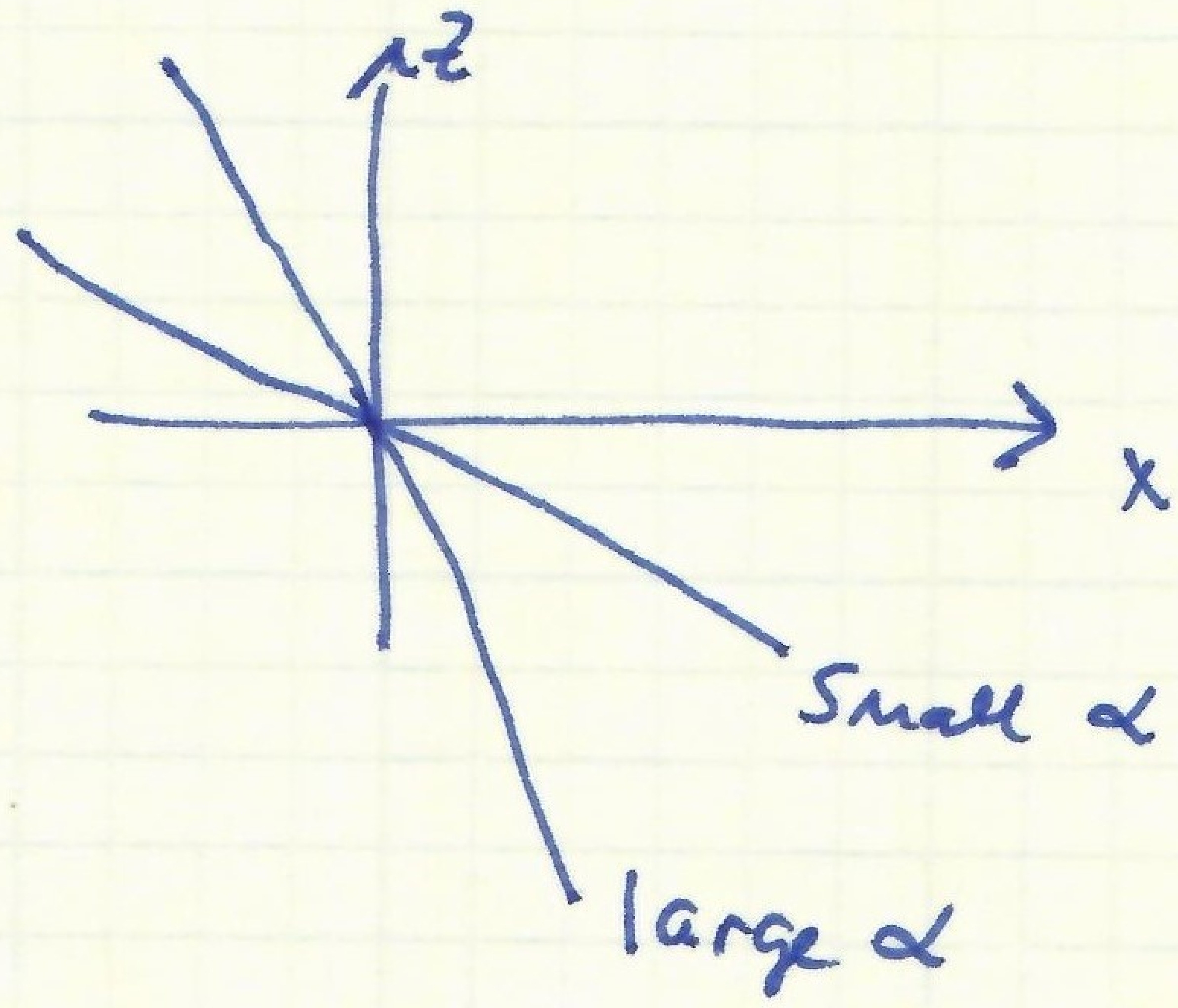
$$\therefore \quad z - z_0 = -\frac{\alpha}{q}(x - x_0)$$




field lines, slope $= -\dfrac{\alpha}{q}$

Since $q = (k^2-\alpha^2)^{1/2}$, increasing $\alpha$ decreases $q$

and $\left|\frac{dz}{dx}\right| = \frac{\alpha}{q}$ increases




Thus, increasing $\alpha$ <u>shears</u> field lines in the $z$ direction

g) field line eq. in x-y plane

$$\frac{dy}{dx} = \frac{B_y}{B_x} = -\frac{k}{q}\tan(kx)$$

$$y-y_0 = -\frac{k}{q}\int_{x_0}^{x} \tan(kx')\,dx'$$

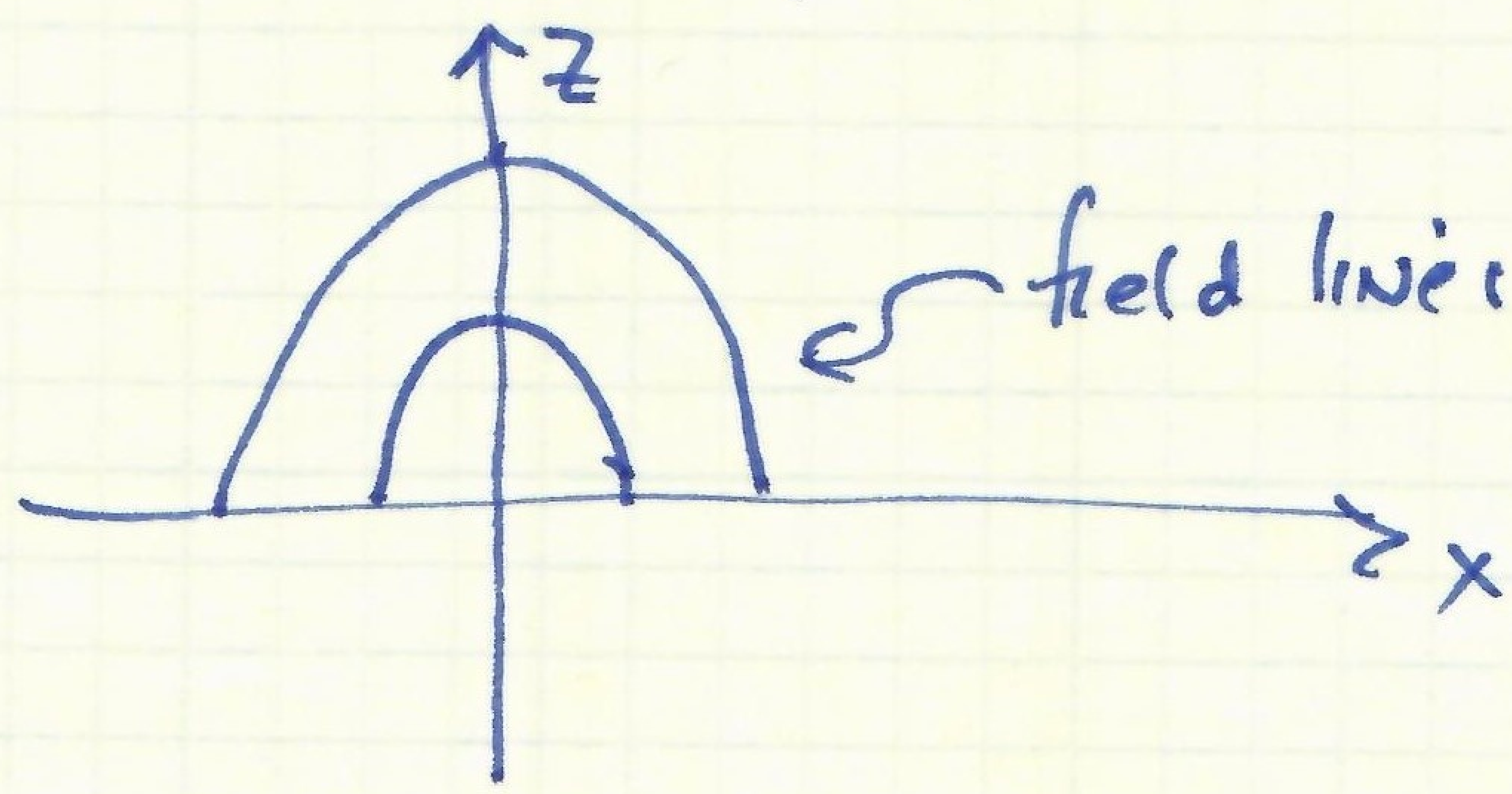
$$= -\frac{1}{q}\int_{kx_0}^{kx} \tan z \, dz \qquad (z = kx')$$

$$= \frac{1}{q}\ln(\cos z)\Big|_{kx_0}^{kx}$$

$$y - y_0 = \frac{1}{q}\left[\ln(\cos kx) - \ln(\cos kx_0)\right]$$

$$= \frac{1}{q}\ln\left(\frac{\cos kx}{\cos kx_0}\right)$$

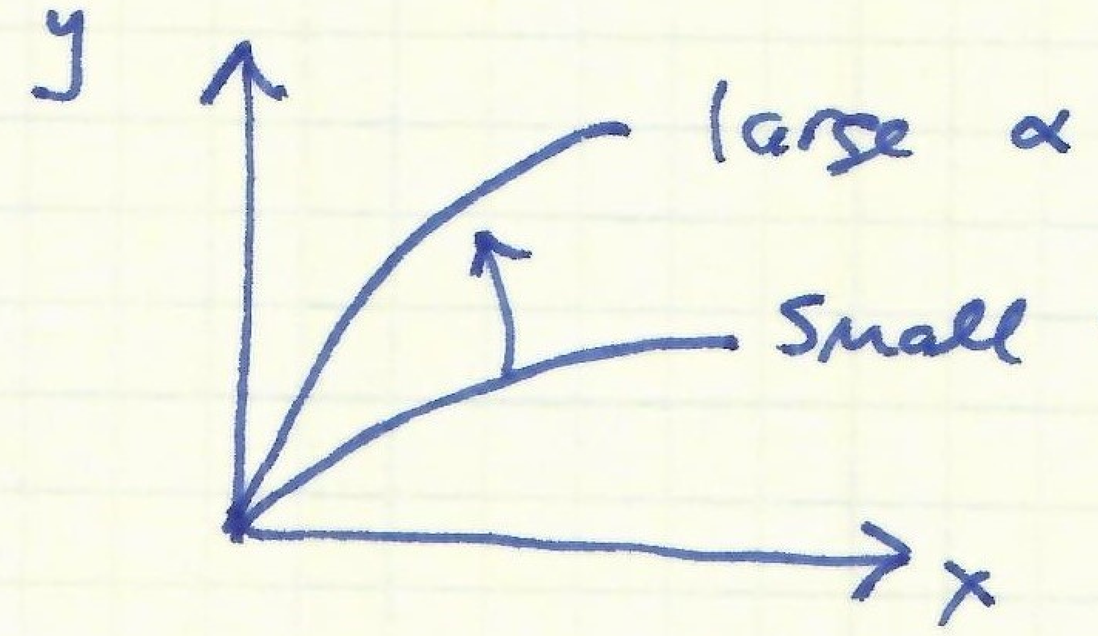
$$y = y_0 + \frac{1}{q}\ln\left(\frac{\cos(kx)}{\cos(kx_0)}\right)$$




h) Note that we had

$$\frac{dy}{dx} = -\frac{k}{q}\tan kx = -\frac{k}{(k^2-\alpha^2)^{1/2}}\tan kx$$

as $\alpha$ is increased, $|dy/dx|$ increases.




Hence, increasing $\alpha$ causes field lines to rise