Solutions to H.W. Set #4

1. Consider a sphere centered at Sun with radius R. The mass crossing this sphere per time is

$$\text{mass crossing per time} = \text{mass flux} \times \text{Surface area}$$

$$= \rho u \times 4\pi R^2$$

Thus, in a time T, the total mass that has crossed the sphere is

$$M_{total} = \rho u \, 4\pi R^2 \, T$$

Here $R = 1\,AU$, the solar wind speed is $400\,km/s$ and the number density $= 5\,cm^{-3}$ (protons).

$$\therefore \quad \rho = (5/3 \times 10^{-24})(5)\ g/cm^3 = \frac{25}{3} \times 10^{-24}\ g/cm^3$$

$$u = 4 \times 10^7\ cm/s$$

$$R = 1.5 \times 10^{13}\ cm$$

$$T = 4.6\ Gy = 1.45 \times 10^{17}\ s$$

$$\Rightarrow \quad \boxed{M_{total} = 1.37 \times 10^{29}\ g = 6.9 \times 10^{-5}\ M_\odot}$$

#2 (a) Recall the MHD momentum eq. in steady state

$$\rho \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} + \nabla P - \frac{1}{c} \underset{\sim}{J} \times \underset{\sim}{B} - \rho \underset{\sim}{g} = 0$$

in this problem, the solar wind is supersonic as we did in class for the Parker spiral so that the ∇P term is small compared to the $\rho \underset{\sim}{u} \cdot \nabla \underset{\sim}{u}$ term. Moreover, $\rho \underset{\sim}{g}$ can also be neglected since it is small compared to $\rho \underset{\sim}{u} \cdot \nabla \underset{\sim}{u}$ where the solar wind has formed.

Thus, we have

$$\underbrace{\rho \underset{\sim}{u} \cdot \nabla \underset{\sim}{u}}_{\nabla \cdot (\rho \underset{\sim}{u}\underset{\sim}{u})} - \frac{1}{4\pi} (\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B} = 0$$

$\nabla \cdot (\rho \underset{\sim}{u}\underset{\sim}{u})$ by using continuity eq. in steady state

$$\Rightarrow \quad \nabla \cdot (\rho \underset{\sim}{u}\underset{\sim}{u}) - \underbrace{\frac{1}{4\pi} (\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B}}_{-\nabla (B^2/8\pi) + \frac{1}{4\pi} \underset{\sim}{B} \cdot \nabla \underset{\sim}{B}} = 0$$

$$\Rightarrow \quad \nabla \cdot \left(\rho \underset{\sim}{u}\underset{\sim}{u} + B^2/8\pi \, \underset{\approx}{I}\right) - \frac{1}{4\pi} \underset{\sim}{B} \cdot \nabla \underset{\sim}{B} = 0$$

note $\quad \underset{\sim}{B} \cdot \nabla \underset{\sim}{B} = \nabla \cdot (\underset{\sim}{B}\underset{\sim}{B}) - \underset{\sim}{B} \, \cancelto{0}{\nabla \cdot \underset{\sim}{B}}$

$$\therefore \quad \nabla \cdot \left(\rho \underset{\sim}{u}\underset{\sim}{u} + B^2/8\pi \, \underset{\approx}{I} - \frac{\underset{\sim}{B}\underset{\sim}{B}}{4\pi}\right) = 0$$

This is the conservative form of the MHD momentum eq. (no pressure, no gravity)

the ϕ component is (note: $\underset{\sim}{u} = u_r \hat{r} + u_\phi \hat{\phi}$, $\underset{\sim}{B} = B_r \hat{r} + B_\phi \hat{\phi}$ no dependence on ϕ)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\rho u_r u_\phi - \frac{B_r B_\phi}{4\pi}\right)\right] + \frac{\rho u_r u_\phi - B_r B_\phi/4\pi}{r} = 0$$

let $f = r^2 u_r u_\phi - B_r B_\phi/4\pi$

$$\Rightarrow \quad \frac{\partial f}{\partial r} = -\frac{f}{r} \Rightarrow f = \frac{c}{r}$$

$$\therefore \quad r^2\left(\rho u_r u_\phi - \frac{B_r B_\phi}{4\pi}\right) = \frac{c}{r}$$

Taking $\boxed{L = r u_\phi - \frac{r B_r B_\phi}{4\pi \rho u_r}}$. we have

$$r \rho u_r L = \frac{c}{r} \Rightarrow L = \frac{c}{r^2 \rho u_r}$$

Since the steady state continuity eq. gives

$\nabla \cdot (\rho \underset{\sim}{u}) = 0 \Rightarrow r^2 \rho u_r = \text{constant}$

$$\Rightarrow \quad \boxed{L = \text{constant}}$$

(b) the steady state magnetic induction equation gives

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{u} \times \underset{\sim}{B}) = 0$$

$$\Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-u_r B_\phi + u_\phi B_r\right)\right] = 0$$

$$\Rightarrow \quad \boxed{-r u_r B_\phi + r u_\phi B_r = \text{constant}}$$

@ the base, $r = R_0$, $B_\phi = 0$, $B_r = B_0$, and

$u_\phi = R_0 \Omega_0 \sin\theta$

$$\Rightarrow \quad \text{constant} = R_0^2 \Omega_0 B_0 \sin\theta$$

thus,

$$\boxed{B_\phi = \frac{u_\phi}{u_r} B_r - \frac{R_0^2 \Omega_0 B_0 \sin\theta}{r u_r}}$$

Inserting this into the boxed equation on page 7, we have

$$r u_\phi = L + \frac{B_r}{4\pi\rho u_r^2}\left[r u_\phi B_r - R_0^2 \Omega_0 B_0 \sin\theta \right]$$

let $M_A = \dfrac{u_r}{v_A} = \dfrac{u_r}{(B_r^2/4\pi\rho)^{1/2}} = \left(\dfrac{4\pi\rho u_r^2}{B_r^2}\right)^{1/2}$

and note $\nabla \cdot \underset{\sim}{B} = 0 \Rightarrow r^2 B_r = R_0^2 B_0$

$$\rightarrow B_r = \left(\frac{R_0}{r}\right)^2 B_0$$

Thus,

$$r u_\phi = L + \frac{1}{B_r M_A^2}\left(r u_\phi B_r - R_0^2 \Omega_0 B_0 \sin\theta\right)$$

$$= L + \frac{1}{M_A^2} r u_\phi - \frac{1}{M_A^2} \frac{R_0^2 \Omega_0 B_0 \sin\theta}{B_r}.$$

$$= L + \frac{1}{M_A^2} r u_\phi - \frac{r^2}{M_A^2} \Omega_0 \sin\theta$$

$$\Rightarrow \quad r u_\phi \left(1 - \frac{1}{M_A^2}\right) = L - \frac{r^2 \Omega_0 \sin\theta}{M_A^2}$$

$$\Rightarrow \quad r u_\phi \left(M_A^2 - 1\right) = L M_A^2 - r^2 \Omega_0 \sin\theta$$

$$r u_\phi = \frac{L M_A^2 - r^2 \Omega_0 \sin\theta}{M_A^2 - 1}$$

$$\boxed{u_\phi = r \Omega_0 \sin\theta \left(\frac{\frac{L}{r^2 \Omega_0 \sin\theta} M_A^2 - 1}{M_A^2 - 1} \right).}$$

This has a singularity at $M_A = 1$ <u>unless</u>

$$\frac{L}{r^2 \Omega_0 \sin\theta} = 1$$ at the point where $M_A = 1$

define this radius as R_A, the Alfven radius

then, $\boxed{L = R_A^2 \Omega_0 \sin\theta}$ R_A is the value of r where $M_A = 1$

note that for $r \gg R_A$, $M_A \gg 1$ and

$$u_\phi = r \Omega_0 \sin\theta \frac{L}{r^2 \Omega_0 \sin\theta} = \frac{L}{r}$$

$\Rightarrow$ $u_\phi \propto 1/r$, as we found before in class.

(c) the total angular momentum loss is

$$\dot{J}_\odot = \underbrace{-\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\, r^2}_{\text{spherical shell}} \underbrace{\rho u_r L \sin\theta}_{\text{angular momentum flux density}}$$

loss

$$= -2\pi r^2 \rho R_A^2 \Omega u_r \int_0^\pi \sin^3\theta\, d\theta$$

$$= -\frac{8\pi}{3} \rho r^2 R_A^2 \Omega_\odot u_r$$

$\dot{M}_\odot$ = solar wind mass loss rate

$$= -\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\, r^2 \underbrace{\rho u_r}_{\text{mass flux}}$$

$$= -4\pi r^2 \rho u_r$$

$$\therefore \quad \dot{J}_\odot = -\frac{8\pi}{3} \rho r^2 R_A^2 \Omega_\odot \left(-\frac{\dot{M}_\odot}{4\pi r^2 \rho}\right)$$

$$\boxed{\dot{J}_\odot = \frac{2}{3} R_A^2 \Omega_\odot \dot{M}_\odot}$$

d) The rate of spin-down is

$$\frac{\dot{J}_o}{J_o} = \frac{2/3\, R_A^2\, \Omega_o \dot{M}_o}{J_o}$$

$$\dot{M}_o = -4\pi \underbrace{r^2 \rho\, u_r}_{\text{constant}}$$

protons

$$= -4\pi\,(1\,AU)^2\,(5 \times 1.6726\times10^{-24}\,g)\,(4\times10^7\,cm/s)$$

$$= 9.4\times10^{11}\ g/s$$

$$\therefore \quad \frac{\dot{J}_o}{J_o} = \frac{(2/3)(8.5\times10^{11})^2(2.8\times10^{-6})(9.4\times10^{11})}{1.9\times10^{48}}$$

$$= 6.7\times10^{-19}\ \frac{rad}{sec}$$

$$\sim 0.1\ \frac{\text{radians}}{4.6\ \text{Billion years}}$$