Solutions to problem set # 5

1. The relevant eq.'s are

$$-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

$$\mathbf{J} = -ne\,\mathbf{u}_e$$

$$n m_e \frac{\partial \mathbf{u}_e}{\partial t} + n m_e\, \mathbf{u}_e \cdot \nabla \mathbf{u}_e = -en\mathbf{E}$$

note that the displacement current is needed because this is an E&M wave. The ions are assumed to be immobile.

Consider small-amplitude perturbation

$$\mathbf{B} = \mathbf{B}'$$

$$\mathbf{E} = \mathbf{E}'$$

$$\mathbf{u}_e = \mathbf{u}_e'$$

$$n = n_0 + n'$$

Insert into above eq.'s, retain only 1^{st}-order, to give

$$\frac{\partial \underset{\sim}{B}'}{\partial t} = -c\nabla \times \underset{\sim}{E}'$$

$$\nabla \times \underset{\sim}{B}' = \frac{4\pi}{c}(-ne\,\underset{\sim}{u}_e') + \frac{1}{c}\frac{\partial \underset{\sim}{E}'}{\partial t}$$

$$n_0 m_e \frac{\partial \underset{\sim}{u}_e'}{\partial t} = -en_0 \underset{\sim}{E}'$$

take $\partial/\partial t$ of the second equation and substitute from the other two to give

$$\nabla \times \frac{\partial \underset{\sim}{B}'}{\partial t} = -\frac{4\pi n_0 e}{c}\frac{\partial \underset{\sim}{u}_e'}{\partial t} + \frac{1}{c}\frac{\partial^2 \underset{\sim}{E}'}{\partial t^2}$$

$$\Rightarrow -c\nabla\times\nabla\times\underset{\sim}{E}' = -\frac{4\pi ne}{c}\left[\frac{-e}{m_e}\underset{\sim}{E}'\right] + \frac{1}{c}\frac{\partial^2 \underset{\sim}{E}'}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 \underset{\sim}{E}'}{\partial t^2} + \underbrace{c^2\nabla\times\nabla\times\underset{\sim}{E}'}_{} = -\frac{4\pi n_0 e^2}{m_e}\underset{\sim}{E}'$$

$$= \nabla(\nabla\cdot\underset{\sim}{E}) - \nabla^2\underset{\sim}{E}$$

but $\nabla\cdot\underset{\sim}{E} = 0$ since $\underset{\sim}{E}$ is transverse to wave propagation direction.

Assume plane-wave solution, to give

$$-\omega^2 + c^2k^2 = \frac{4\pi n_0 e^2}{m_e} = -\omega_e^2$$

$$\Rightarrow \boxed{\omega^2 = \omega_e^2 + c^2k^2}$$

#2 a. Taking

$$\underset{\sim}{\delta v} = V_g \cos\omega t\,(\hat{x}\cos kx \sin ky - \hat{y}\sin kx \cos ky)$$

we note

$$\nabla\cdot\underset{\sim}{\delta v} = V_g \cos\omega t\left[-k\sin kx \sin ky + k\sin kx \sin ky\right]$$

$$= 0$$

Thus, the continuity equation gives

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\,\underset{\sim}{\delta v}) = \frac{\partial \rho}{\partial t} + \rho\cancelto{0}{\nabla\cdot\underset{\sim}{\delta v}} + \underset{\sim}{\delta v}\cdot\nabla\rho = 0$$

$$\Rightarrow \quad \frac{D}{Dt}\rho = 0$$

$\rho =$ constant along flow streamlines

But all stream lines start at base, and at the base, $\rho =$ cons. Thus, <u>ρ must be constant everywhere.</u>

b. The momentum equation gives

$$\rho_0 \frac{\partial \underset{\sim}{\delta v}}{\partial t} + \rho_0 \cancelto{0}{\underset{\sim}{\delta v}\cdot\nabla\underset{\sim}{\delta v}} = \frac{1}{c}\underset{\sim}{J}\times\underset{\sim}{B}$$

(second-order)

$$= \frac{1}{4\pi}(\nabla\times\underset{\sim}{\delta B})\times\underset{\sim}{B}_0$$

$$\Rightarrow \quad \boxed{\rho_0 \frac{\partial \underset{\sim}{\delta v}}{\partial t} = \frac{B_0}{4\pi}\frac{\partial \underset{\sim}{\delta B}}{\partial z}} \qquad (1)$$

The induction equation gives

$$\frac{\partial \delta \underset{\sim}{B}}{\partial t} = \nabla \times (\delta \underset{\sim}{V} \times \underset{\sim}{B}_0)$$

$\vdots$

$$\boxed{\frac{\partial \delta \underset{\sim}{B}}{\partial t} = B_0 \frac{\partial \delta \underset{\sim}{v}}{\partial z}} \qquad (2)$$

Taking $\partial/\partial t$ (1) and subst from (2) gives

$$\boxed{\frac{\partial^2 \delta \underset{\sim}{v}}{\partial t^2} = v_A^2 \frac{\partial^2 \delta \underset{\sim}{v}}{\partial z^2} \quad ; \quad v_A^2 = \frac{B_0^2}{4\pi \rho_0}}$$

These are Shear Alfven waves

c. Assume plane-wave solution of the form

$$\delta \underset{\wedge}{v} = \delta \underset{\wedge}{v}' \cos(\omega t - kz) \qquad \text{(only upward moving)}$$

@ $z=0$ $\delta \underset{\sim}{v} = V_g \cos\omega t \,(\cos kx \sin ky \,\hat{x} - \sin kx \cos ky \,\hat{y})$

$\Rightarrow$ $\delta \underset{\sim}{v}' = V_g \cos\omega t \,(\cos kx \sin ky \,\hat{x} - \sin kx \cos ky \,\hat{y})$

$$\therefore \boxed{\delta \underset{\sim}{v} = V_g \cos(\omega t - kz)(\cos kx \sin ky \,\hat{x} - \sin kx \cos ky \,\hat{y})}$$

Also, because these are shear Alfven waves

$$\delta \mathbf{B} = -B_0 \frac{\delta \mathbf{v}}{V_A}$$

$$\Rightarrow \boxed{\delta \mathbf{B} = -B_0 \frac{V_g}{V_A} \cos(\omega t - kz)(\cos kx \sin ky \, \hat{x} - \sin kx \cos ky \, \hat{y})}$$

d. The wave kinetic energy is

$$U_w = \frac{1}{2} \rho_0 \langle \delta v^2 \rangle \qquad \delta v^2 = \delta \mathbf{v} \cdot \delta \mathbf{v}$$

$\langle \rangle \Rightarrow$ over one period

$$= \frac{1}{2} \rho_0 V_g^2 \alpha^2 \langle \cos^2(\omega t - kz) \rangle$$

where $\alpha^2 = \cos^2 kx \sin^2 ky + \sin^2 kx \cos^2 ky$

and the magnetic energy is

$$U_B = \frac{\langle \delta B^2 \rangle}{8\pi} \qquad \delta B^2 = \delta \mathbf{B} \cdot \delta \mathbf{B}$$

$$= \underbrace{B_0^2 \frac{1}{8\pi V_A^2}}_{\frac{1}{2}\rho_0} V_g^2 \alpha^2 \langle \cos^2(\omega t - kz) \rangle$$

$$= \frac{1}{2} \rho_0 V_g^2 \alpha^2 \langle \cos^2(\omega t - kz) \rangle = U_w$$

e. $$\underset{\sim}{F}_w = \frac{c}{4\pi}\,\delta\underset{\sim}{E} \times \delta\underset{\sim}{B}$$

$$= -\frac{1}{4\pi}\left(\delta\underset{\sim}{v} \times \underset{\sim}{B}_0\right) \times \delta\underset{\sim}{B}$$

$$= -\frac{1}{4\pi}\left(\delta\underset{\sim}{v} \cdot \delta\underset{\sim}{B}\right) B_0\,\hat{z}$$

$$\Rightarrow \quad |\underset{\sim}{F}_w| = \frac{B_0}{4\pi}\langle \delta\underset{\sim}{v} \cdot \delta\underset{\sim}{B} \rangle$$

$$= \frac{B_0}{4\pi} V_g^2 B_0 \frac{1}{V_A} \alpha^2 \underbrace{\langle \cos^2(\omega t - kz) \rangle}_{1/2}$$

$$= \frac{B_0^2}{8\pi V_A} V_g^2 \alpha^2$$

$$\boxed{|\underset{\sim}{F}_w| = \frac{1}{2}\rho_0 V_A V_g^2 \alpha^2}$$

f. using #'s, we find (take $\alpha = 1 \rightarrow$ max value)

$$V_A = \frac{B_0}{(4\pi\rho_0)^{1/2}} = \frac{100}{(4\pi \times 10^{-15})^{1/2}} \frac{cm}{s} = 8.92 \times 10^8 \frac{cm}{s}$$

$$\Rightarrow \quad |\underset{\sim}{F}_w| = \frac{1}{2}(10^{-15})(8.92\times10^8)(6\times10^4)^2(1) \quad \frac{erg}{cm^2 \cdot s}$$

$$\boxed{|\underset{\sim}{F}_w| = 1.6 \times 10^7 \; erg/cm^2 \cdot s}$$ < 1% of total outward flux

\# 3. a) The problem states we are only dealing with waves along $\mathbf{B}_0$, and we have a cold plasma. Thus perturbations in $\mathbf{B}$ and $\mathbf{E}$ will be transverse to $\mathbf{B}_0$ and, hence, transverse to $\mathbf{k}$. Thus, $\nabla \cdot \mathbf{E} = 0$! Thus the density remains constant. Thus the relevant equations are

$$m_e n_0 \frac{\partial \mathbf{u}_e}{\partial t} + m_e n_0 \mathbf{u}_e \cdot \nabla \mathbf{u}_e = -e n_0 \mathbf{E} - \frac{e n_0}{c} \mathbf{u}_e \times \mathbf{B}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = -\frac{4\pi n e}{c} \mathbf{u}_e + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}$$

the last one assumes $\mathbf{J} = n_0 e(\cancelto{0}{\mathbf{u}_i} - \mathbf{u}_e) = -n_0 e \mathbf{u}_e$

perturb with

$$\mathbf{u}_e = \mathbf{u}_e'$$

$$\mathbf{B} = \mathbf{B}' + \mathbf{B}_0$$

$$\mathbf{E} = \mathbf{E}'$$

and we obtain

$$m_e n_0 \frac{\partial \underset{\sim}{u}_e'}{\partial t} = -e n_0 \underset{\sim}{E}' - \frac{e n_0}{c} \underset{\sim}{u}_e' \times \underset{\sim}{B}_0$$

$$\nabla \times \underset{\sim}{E}' = -\frac{1}{c} \frac{\partial \underset{\sim}{B}'}{\partial t}$$

$$\nabla \times \underset{\sim}{B}' = -\frac{4\pi n_0 e}{c} \underset{\sim}{u}_e' + \frac{1}{c} \frac{\partial \underset{\sim}{E}'}{\partial t}$$

b). Assume plane wave solution to give

$$-i\omega m_e n_0 \underset{\sim}{u}_e' = -e n_0 \underset{\sim}{E}' - \frac{e n_0}{c} \underset{\sim}{u}_e' \times \underset{\sim}{B}_0$$

$$i \underset{\sim}{k} \times \underset{\sim}{E}' = \frac{1}{c} i\omega \underset{\sim}{B}'$$

$$i \underset{\sim}{k} \times \underset{\sim}{B}' = -\frac{4\pi n_0 e}{c} \underset{\sim}{u}_e' - \frac{i\omega}{c} \underset{\sim}{E}'$$

c) $\underset{\sim}{k} = k\hat{z}$, $\underset{\sim}{B}_0 = B_0 \hat{z}$

and, since there is no component of $\underset{\sim}{E}'$ along $\underset{\sim}{B}_0$ we have $u_{ez}' = 0$, $E_z' = 0$

This leaves

$$\boxed{\begin{aligned}
-i\omega m_e n_0 u'_{ex} &= -e n_0 E'_x - \frac{e}{c} n_0 u'_{ey} B_0 \\
-i\omega m_e n_0 u'_{ey} &= -e n_0 E'_y + \frac{e}{c} n_0 u'_{ex} B_0 \\
-ik E'_y &= i\frac{\omega}{c} B'_x \\
ik E'_x &= i\frac{\omega}{c} B'_y \\
-ik B'_y &= -\frac{4\pi n_0 e}{c} u'_{ex} - i\frac{\omega}{c} E'_x \\
ik E'_x &= -\frac{4\pi n_0 e}{c} u'_{ey} - i\frac{\omega}{c} E'_y
\end{aligned}}$$

d.) See my solution on these waves contained in the set of notes for this class, the dispersion relation turns out to be

$$\frac{\omega^2 - k^2c^2}{\omega_e^2}\left(1 \mp \frac{\Omega_e}{\omega}\right) = 1$$

(See also eq. 12.96 in Kivelson & Russell)