PTYS 558 2/19/2020

Potential and Force-free fields

recall the MHD momentum eq.

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \frac{1}{c} \mathbf{J} \times \mathbf{B} - \rho \mathbf{g}$$

static means ~~no flow~~ time dependence ($\cancel{\mathbf{u}=0}$ $\partial/\partial t \to 0$)

stationary means no flow ($\mathbf{u} = 0$)

ratio of ∇P & $\frac{1}{c} \mathbf{J} \times \mathbf{B}$ terms $\propto \beta$

if $\beta \ll 1 \Rightarrow \frac{1}{c} \mathbf{J} \times \mathbf{B}$ term dominates

also, even if there is some flow, $\mathbf{u}$, near the Sun $\rho u^2 \ll B^2/8\pi$, for example

thus, only $1/c\, \mathbf{J} \times \mathbf{B}$ is important

$$\frac{1}{c} \mathbf{J} \times \mathbf{B} = 0$$

case ① $\mathbf{J} = 0 \Rightarrow$ no current $\rightarrow$ potential field

② $\mathbf{J} \parallel \mathbf{B} \rightarrow$ force-free consider case (1) $\underset{\sim}{J} = 0 \rightarrow \nabla \times \underset{\sim}{B} = 0$

$\underset{\sim}{B} = \nabla \psi_B$ $\quad \psi_B =$ scalar magnetic potential

because $\nabla \cdot \underset{\sim}{B} = 0$

$\Rightarrow \quad \nabla^2 \psi_B = 0$ Laplaces equation

in spherical coordinates, laplaces eq. is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi_B}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi_B}{\partial \theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi_B}{\partial \phi} = 0$$

assume $\psi_B = R(r)\,\Theta(\theta)\,\Phi(\phi)$

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

$$\underset{\text{set}}{=} m^2$$

$\therefore$ 1 eq. is $\dfrac{d^2\Phi}{d\phi^2} = -\Phi m^2$

$$\Rightarrow \quad \boxed{\Phi = e^{im\phi}} \qquad m = 0, 1, 2 \ldots$$

the other eq., after dividing by $\sin^2\theta$ is

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{m^2}{\sin^2\theta}$$

$$\underset{\text{set}}{=} n(n+1)$$

resulting eq.'s are

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = R\, n(n+1)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \left[\frac{m^2}{\sin^2\theta} - n(n+1)\right]\Theta = 0$$

the solutions are

$$R = a_n' r^n + b_n' r^{-(n+1)}$$

$$\Theta = P_n^m(\cos\theta) \qquad \text{associated Legendre polynomials}$$

$$n \to 0, 1, 2, \ldots$$

$$m \to -n, \ldots, n$$

if field is dipolar at the photosphere

$\Rightarrow \quad B_r(r = R_\odot) = 2 B_\odot \cos\theta$ "dipolar field"

define the "source surface"

$$B_\theta(r = R_s) = 0$$

these are the 2 B.C.'s

use these B.C.'s, solve for a & b, and the final solution becomes

$$\boxed{\begin{aligned} B_r &= 2B_\odot \frac{1 + 2(R_s/r)^3}{1 + 2(R_s/R_\odot)^3} \cos\theta \\ B_\theta &= -2B_\odot \frac{1 - (R_s/r)^3}{1 + 2(R_s/R_\odot)^3} \sin\theta \end{aligned}}$$

to get magnetic field lines

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} = \frac{ds}{B} \qquad \text{(cartesian)}$$

$B = |\underset{\sim}{B}|$

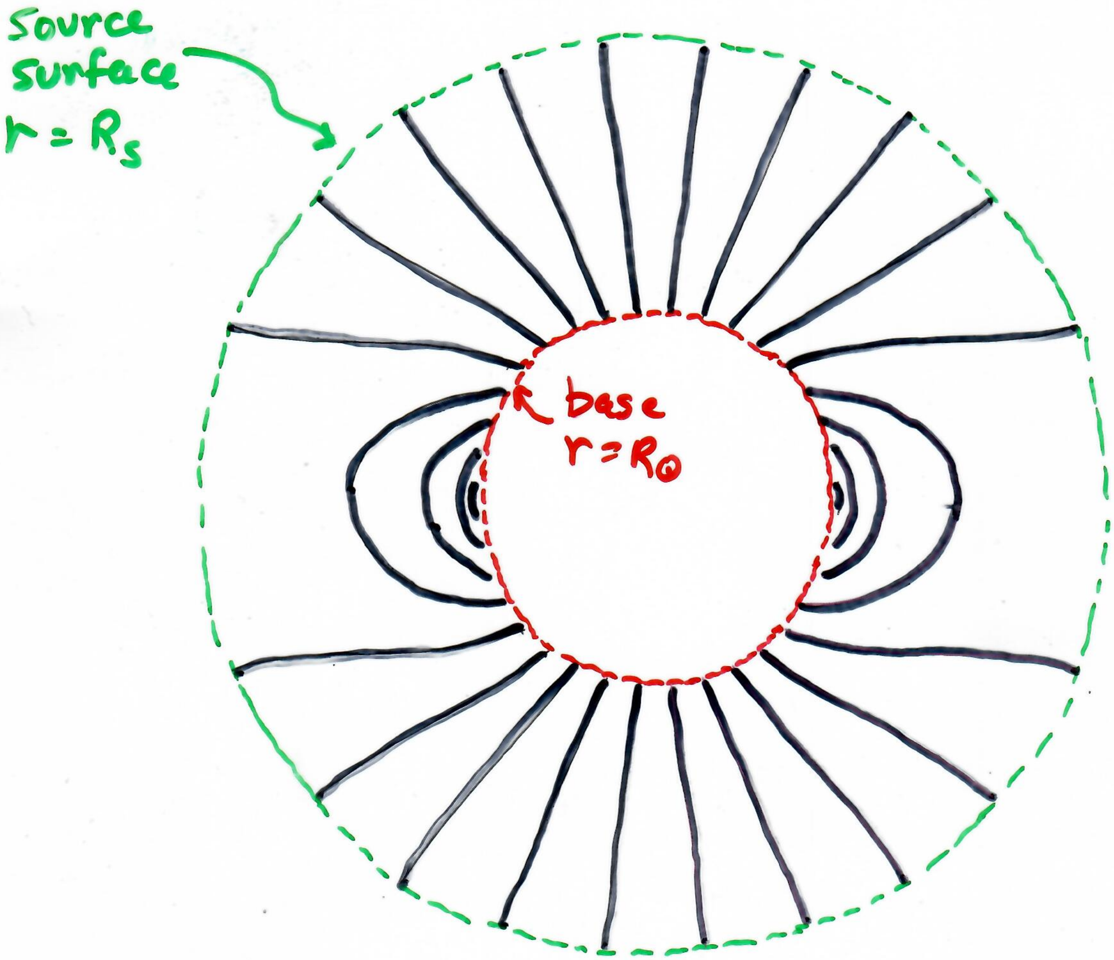
s is along $\underset{\sim}{B}$

$$\frac{dr}{B_r} = \frac{r\,d\theta}{B_\theta} = \frac{r\sin\theta\,d\phi}{B_\phi} = \frac{ds}{B} \qquad \text{(spherical)}$$

for our case, we would use

$$\frac{dr}{B_r(r,\theta)} = \frac{r\,d\theta}{B_\theta(r,\theta)}$$

Simple Potential Field Model



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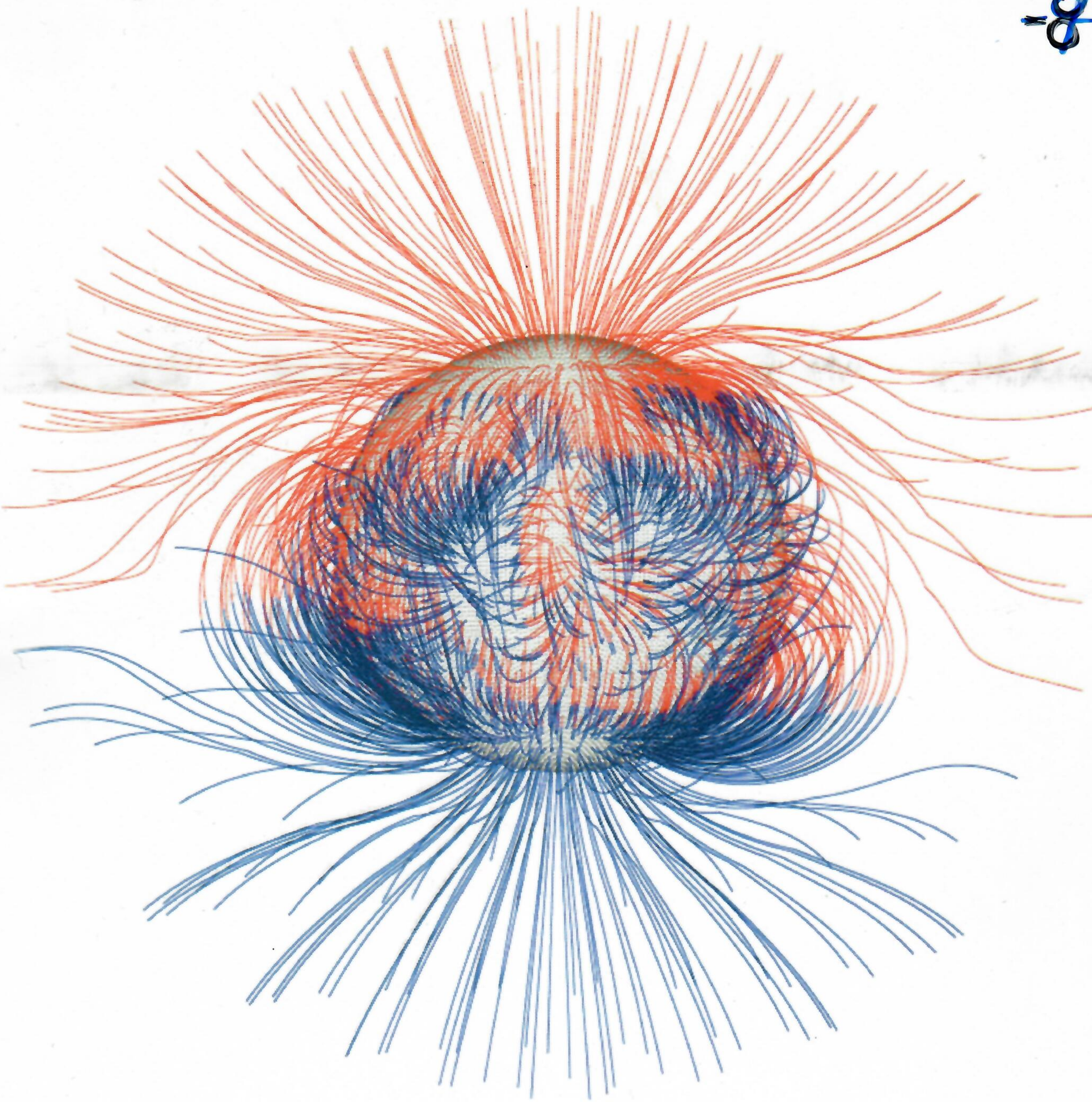

Potential Field Source Surface

Case ② $\underset{\sim}{J} \parallel \underset{\sim}{B}$

Consider $\underset{\sim}{J} = \frac{c}{4\pi} \nabla \times \underset{\sim}{B} = \frac{c}{4\pi} \alpha \underset{\sim}{B}$

set

$\therefore \quad \nabla \times \underset{\sim}{B} = \alpha \underset{\sim}{B} \qquad \alpha =$ scalar function of position (or a constant)

note ~~$\nabla \cdot \underset{\sim}{B}$~~ $\nabla \cdot (\nabla \times \underset{\sim}{B}) = 0$

$$= \nabla \cdot (\alpha \underset{\sim}{B})$$

$$= \alpha \cancelto{0}{\nabla \cdot \underset{\sim}{B}} + \underset{\sim}{B} \cdot \nabla \alpha$$

$$\Rightarrow \quad \underset{\sim}{B} \cdot \nabla \alpha = 0$$

$\Rightarrow \quad \alpha =$ constant along magnetic field lines

A simple example:

$$\underset{\sim}{B} = (0, B_y(x), B_z(x))$$

1D, and $\nabla \cdot \underset{\sim}{B} = 0$

$$\nabla \times \underset{\sim}{B} = \begin{array}{c} \begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \end{array} \\ \begin{vmatrix} \partial/\partial x & 0 & 0 \\ 0 & B_y & B_z \end{vmatrix} \end{array} = -\frac{\partial B_z}{\partial x}\hat{y} + \frac{\partial B_y}{\partial x}\hat{z}$$

$$\underset{\text{set}}{=} \alpha \underset{\sim}{B}$$

$$\Rightarrow \quad -\frac{\partial B_z}{\partial x} = \alpha B_y$$

$$\frac{\partial B_y}{\partial x} = \alpha B_z$$

$$\Rightarrow \quad B = (B_y^2 + B_z^2)^{1/2} = \text{constant}$$

solution is

$$B_y = \pm B_0 \sin(\alpha x)$$

$$B_z = \pm B_0 \cos(\alpha x)$$

corkscrew pattern const. $|\underset{\sim}{B}|$

more generally, if we have a 2D problem

$$\underset{\sim}{B} = \left(B_x(x,y),\ B_y(x,y),\ B_z(x,y)\right)$$

"2D magnetostatics"

define A = $\hat{z}$ component
vector potential

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A} = \nabla \times A\hat{z}$$

it can be shown ... A is constant
along field lines

In fact A defines a field line!

it can also be shown (even if P (thermal pressure) is
included in the analysis) and gravity ignored)

$$P + \frac{B_z^2}{8\pi} = \text{constant along field lines}$$

also,

$$P = \text{constant along field lines}.$$

$$\therefore \; B_z = \text{“} \quad \text{“} \quad \text{“} \quad \text{“}$$

P & B_z are functions of A only.
This leads to a useful eq. in 2D magnetostatics

$$\boxed{\nabla^2 A + 4\pi \frac{d}{dA}\left(P + \frac{B_z^2}{8\pi}\right) = 0}$$

Grad-Shafranov
equation