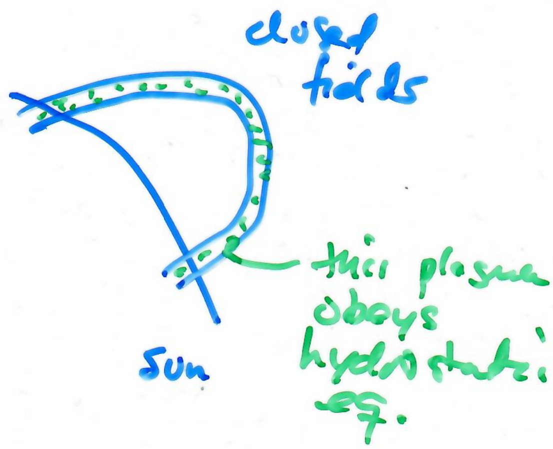
Hydrostatic Equilibrium in a Stellar or Planetary Atmosphere

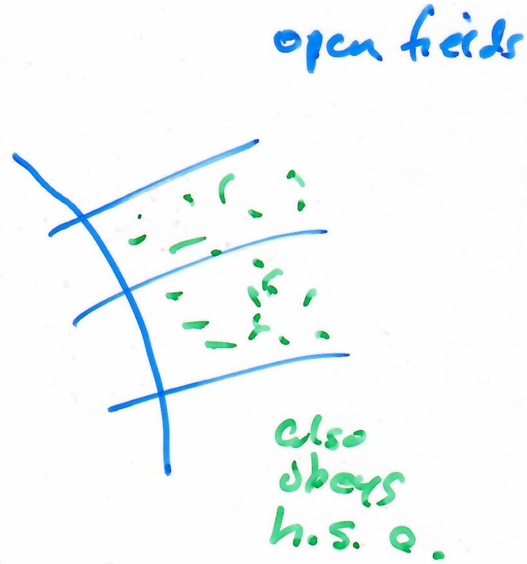
Consider the MHD momentum equation, steady state, and static ($\underset{\sim}{u} = 0$), LHS is 0

$$0 = -\nabla P + \frac{1}{c}\, \underset{\sim}{J} \times \underset{\sim}{B} - \rho \underset{\sim}{g}$$

along field lines (i.e. dot $\underset{\sim}{B}$ into this eq.), one have

$$-\nabla P - \rho \underset{\sim}{g} = 0 \qquad \text{eq. of hydrostatic equil.}$$





Consider a star, or the Sun, planet, . a "spherical" atmosphere

$$-\frac{\partial P}{\partial r} = \rho \frac{G M_\odot}{r^2} \qquad \text{in } \hat{r} \text{ direction}$$

we also have the ideal gas law

$$P = nkT$$

$$= \frac{\rho}{m} kT \qquad m = \text{mean mass of particles}$$

$$\simeq m_p$$

$$\Rightarrow \rho = \frac{m_p P}{kT}$$

$$\therefore -\frac{dP}{dr} = \frac{m_p G M_\odot}{kT} \frac{1}{r^2} P$$

what is T? It is reasonable to consider

$$T = T_0 \left(\frac{R_0}{r}\right)^\alpha$$

$\alpha = 0$ isothermal

$\alpha = 2/7$ based on Spitzer conductivity due to electron heat flux

$\alpha < 0$ probably unphysical

$$\Rightarrow \frac{dP}{dr} = -\frac{m_p G M_\odot}{k T_0 (R_0/r)^\alpha} \frac{1}{r^2} P$$

$$= \frac{-m_p G M_\odot}{k T_0} \left(\frac{r}{R_0}\right)^\alpha \left(\frac{R_0}{r}\right)^2 \frac{1}{R_0^2} P$$

$$= \frac{-m_p G M_\odot}{k T_0 R_0^2} \left(\frac{r}{R_0}\right)^{\alpha - 2} P$$

$$= -\frac{1}{H}\left(\frac{r}{R_0}\right)^{\alpha-2} p$$

$$H = \text{atmospheric scale height} = \frac{k T_0 R_0^2}{m_p G M_\odot} = \frac{k T_0}{m g}$$

g = acc. due to grav. at $r = R_0$

$$\therefore \quad \frac{dP}{P} = -\frac{1}{H}\left(\frac{r}{R_0}\right)^{\alpha-2} dr$$

$\int$ be a bit careful! $\alpha = 1$ case is different

$$\ln P = -\frac{1}{H}\frac{1}{\alpha-1}\left(\frac{r}{R_0}\right)^{\alpha-1} R_0 + C$$

$\vdots$

$$\boxed{\begin{aligned} P(r) &= P_0\, e^{-\frac{R_0}{H}\frac{1}{\alpha-1}\left[\left(\frac{r}{R_0}\right)^{\alpha-1} - 1\right]} && \alpha \neq 1 \\ p(r) &= P_0\left(\frac{r}{R_0}\right)^{-\frac{R_0}{H}} && \alpha = 1 \end{aligned}}$$

if $\alpha \neq 1$ $(\alpha < 1)$, then as $r \to \infty$

$$P(r) \to P_0\, e^{\frac{R_0}{H}\frac{1}{\alpha-1}}$$ unphysical!

not $\alpha = 0 \Rightarrow P(r) \to P_0 e^{-R_0/H} \neq 0$!

For plasmas heat flux is due to electrons

$$\underset{\sim}{Q} = K \nabla T$$

$$K = C T^{5/2}$$ for electrons

from Spitzer, 1962

$\nabla \cdot \underset{\sim}{Q} = 0$, we get

$$T \propto r^{-2/7}$$

leads to

$$P = P_0 e^{-\frac{7}{5}\frac{r_0}{H}} \neq 0 \;!$$

<u>Consider a wind solution instead</u>

Parker, 1958

recall momentum eq

$$\rho \frac{\partial \underset{\sim}{u}}{\partial t} + \rho \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} = -\nabla P + \frac{1}{c} \underset{\sim}{J} \times \underset{\sim}{B} - \rho \underset{\sim}{g}$$

steady state, no $\underset{\sim}{B}$, ~~this is~~ (radial only) spherically symmetric

we have

$$\boxed{\rho u \frac{du}{dr} = -\frac{dP}{dr} - \rho \frac{G M_\odot}{r^2}} \qquad (1)$$

other necessary equations are

$$\nabla \cdot (\rho \underset{\sim}{u}) = 0 \Rightarrow \boxed{\rho u r^2 = \text{constant}} \quad (2)$$

also, can assume adiabatic atmosphere

$$P \propto \rho^{\gamma}$$

if $\gamma = 1 \Rightarrow$ isothermal atmosphere

$$\therefore \boxed{\frac{P}{\rho} = \text{constant}}$$

define $M = \begin{array}{c}\text{Mach}\\ \text{number}\end{array} = \dfrac{u}{C_s}$

where C_s = sound speed $= \left(\dfrac{P}{\rho}\right)^{1/2}$ isothermal

$\left(C_s = \left(\dfrac{\gamma P}{\rho}\right)^{1/2} \text{ generally}\right)$

divide (1) by C_s^2

$$\frac{\rho u}{C_s^2}\frac{du}{dr} = -\frac{1}{C_s^2}\frac{dP}{dr} - \frac{\rho g}{C_s^2} \qquad g = \frac{GM_\odot}{r^2}$$

also $$P = \rho c_s^2$$

$$dP = d\rho\, c_s^2$$

$$\therefore \quad \frac{\rho u}{c_s^2}\frac{du}{dr} = -\frac{d\rho}{dr} - \frac{\rho g}{c_s^2}$$

$$\Rightarrow \quad \frac{\rho}{u} M^2 \frac{du}{dr} = -\frac{d\rho}{dr} - \frac{\rho g}{c_s^2}$$

$$\Rightarrow \quad M^2 \frac{du}{u} = -\frac{d\rho}{\rho} - \frac{g}{c_s^2}dr$$

$$= -\frac{d\rho}{\rho} - \frac{GM_\odot}{c_s^2}\frac{dr}{r^2}$$

since c_s = constant $\quad M = u/c_s \Rightarrow du = c_s\, dM$

$$\Rightarrow \quad M^2 \frac{dM}{M} = -\frac{d\rho}{\rho} - \frac{GM_\odot}{c_s^2}\frac{dr}{r^2} \qquad (3)$$

and from (2)

$\rho u r^2$ = const. $\Rightarrow d\rho\, u r^2 + \rho\, du\, r^2 + \rho u\, 2r\, dr = 0$

⋮

$$2\frac{dr}{r} + \frac{d\rho}{\rho} + \frac{dM}{M} = 0 \qquad (4)$$

Combine (3) & (4), gives

$$M^2 \frac{dM}{M} = 2\frac{dr}{r} + \frac{dM}{M} - \frac{GM_\odot}{c_s^2}\frac{dr}{r^2}$$

now define $\boxed{r_c = \frac{GM_\odot}{2c_s^2}}$

and $r' = r/r_c$

$dr' = dr'/r_c$

we get

$$\frac{dM}{dr'}\left(M - \frac{1}{M}\right) = \frac{2}{r'} - \frac{2}{r'^2}$$

integrate to give

$$\frac{1}{2}M^2 - \ln M = 2\ln r' + \frac{2}{r'} + C^*$$

final solution

$$\boxed{M^2 - \ln M^2 = 4 \ln\left(\frac{r}{r_c}\right) + 4\left(\frac{r_c}{r}\right) + C^*}$$

$\frac{r_c}{r}$