Parker-spiral field w/ radial $\underset{\sim}{B}$ at source surface

Recall, we had

$$B_r = B_0 (R_0/r)^2$$
$$B_\phi = - B_0 R_0/r \frac{R_0 \Omega_0 \sin\theta}{V_0}$$
$\Big\}$ this is correct.

at $r = R_0$, $B_\phi \neq 0$.

Consider now having $B_\phi = 0$ @ $r = R_0$

Recall the boundary calculation we did. Apply again

$$\underset{\sim}{E}\,(r = R_0) = -\frac{1}{c}\, \underset{\sim}{v} \times \underset{\sim}{B} = -\frac{1}{c} \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ 0 & 0 & V_\phi \\ B_r & 0 & 0 \end{vmatrix}$$

$$= -\frac{1}{c} B_r V_\phi \,\hat{\theta}$$

$$\underset{\sim}{E}\,(r = R_0 + \epsilon) = -\frac{1}{c} \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ V_0 & 0 & V_\phi \\ B_r & 0 & 0 \end{vmatrix}$$

$$= -\frac{1}{c} B_r V_\phi \,\hat{\theta}$$

continuous; so this is OK

thus, a better kinematic flow field is

$$\underset{\sim}{V} = \begin{cases} V_\phi \hat{\phi} & ; r = R_0 \\ V_0 \hat{r} + V_\phi(r)\,\hat{\phi} & ; r > R_0 \end{cases}$$

lets also assume V_0 = constant.

Parker took V_ϕ = constant violating angular momentum cons.

see also Cravens, Parks

other books are correct Boyd & Sanderson

also Priest

Consider the ϕ-component of the MHD momentum eq. (steady state)

$$\left(\underset{\sim}{u} \cdot \nabla \underset{\sim}{u}\right)_\phi = \left(-\nabla P\right)_\phi + \left(\frac{1}{c}\, \underset{\sim}{J} \times \underset{\sim}{B}\right)_\phi - \left(\rho \underset{\sim}{g}\right)_\phi$$

vector identity

$$\left(\underset{\sim}{A} \cdot \nabla \underset{\sim}{B}\right)_\phi = A_r \frac{\partial B_\phi}{\partial r} + \frac{A_\theta}{r} \frac{\partial B_\phi}{\partial \theta} + \frac{A_\phi}{r \sin\theta} \frac{\partial B_\phi}{\partial \phi} + \frac{A_\phi B_r}{r} + \frac{\cot\theta\, A_\phi B_\theta}{r}$$

$$\therefore \quad u_r \frac{\partial u_\phi}{\partial r} + \cancelto{0}{\frac{u_\theta}{r} \frac{\partial u_\phi}{\partial \theta}} + \cancelto{0}{\frac{u_\phi}{r \sin\theta} \frac{\partial u_\phi}{\partial \phi}} + \frac{u_\phi u_r}{r} + \cancelto{0}{\frac{\cot\theta\, u_\phi u_\theta}{r}}$$

$$= -\cancelto{0}{\frac{\partial p}{r \partial \phi}} + 0 + 0$$

ignore Mag. stresses

no grav. variation in ϕ

$$\Rightarrow \quad u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\phi u_r}{r} = 0$$

$$\therefore \quad \frac{\partial u_\phi}{\partial r} = -\frac{u_\phi}{r}$$

$u_\phi =$ const is <u>not</u> a solution

Can solve this, and we find

$$u_\phi = u_\phi (R_0) \frac{R_0}{r}$$

$$\boxed{u_\phi = R_0 \Omega_0 \sin\theta \frac{R_0}{r}} \qquad (1)$$

we now solve mag. inductn eq. using this form of u_ϕ (V_ϕ)

$$\underset{\sim}{V} = \begin{cases} R_0 \Omega_0 \sin\theta \, \hat{\phi} & r = R_0 \\ V_0 \hat{r} + R_0 \Omega_0 \sin\theta \frac{R_0}{r} \hat{\phi} & r > R_0 \end{cases}$$

solve $\cancel{\frac{\partial \underset{\sim}{B}}{\partial t}}^{\to 0} = \nabla \times (\underset{\sim}{V} \times \underset{\sim}{B}) = 0$

solve for $\underset{\sim}{B}$

$$\underset{\sim}{V} \times \underset{\sim}{B} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ V_0 & 0 & V_\phi \\ B_r & 0 & B_\phi \end{vmatrix}$$

$$= (-V_0 B_\phi + V_\phi B_r) \hat{\theta}$$

$$\nabla \times (\underset{\sim}{V} \times \underset{\sim}{B}) = \frac{1}{r} \frac{\partial}{\partial r} \left[r(-V_0 B_\phi + V_\phi B_r) \right] \hat{\phi} = 0$$

$$\therefore \quad -r V_0 B_\phi + r V_\phi B_r = \text{constant}$$

Also, $\nabla \cdot \underset{\sim}{B} = 0$

$$\Rightarrow \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 B_r) = 0 \qquad \text{(no } \phi \text{ dependence)}$$

$$\therefore \quad \boxed{B_r = B_0 \left(R_0/r\right)^2}$$

@ $r = R_0$, we have

$$\cancel{r} \quad R_0 V_\phi(R_0) B_0 = \text{constant}$$

$$\therefore \quad -r V_0 B_\phi + r V_\phi B_r = R_0 V_\phi(R_0) B_0$$

$$\Rightarrow \quad -r V_0 B_\phi + r\left(R_0 \Omega_0 \sin\theta \frac{R_0}{r}\right) B_0 \left(\frac{R_0}{r}\right)^2 = R_0^2 \Omega_0 \sin\theta \, B_0$$

$$\vdots$$

$$\Rightarrow \quad \boxed{B_\phi = -B_0 \frac{R_0}{r} \frac{R_0 \Omega_0 \sin\theta}{V_0}\left[1 - \left(\frac{R_0}{r}\right)^2\right]}$$

~~[illegible]~~

Parker spiral with fluctuations added

"WKB" limit → small scale variations in $\underset{\sim}{B}$ due to plasma waves

"quasi-static" limit → large scale variations in $\underset{\sim}{B}$ due to random motions of the magnetic footpoints

(~~Par~~ Jokipii, Parker, 1969)

consider the ~~net~~ variation of $\underset{\sim}{B}$ due to "wiggling" of magnetic footpoints at the Sun (or source surface)

this is a time-dependent problem

still kinematic

i.e.

$$\underset{\sim}{V} = \begin{cases} V_\theta(\theta,\phi)\,\hat{\theta} + (R_0\Omega_0\sin\theta + V_\phi')\,\hat{\phi} & r = R_0 \\ V_0\,\hat{r} & r > R_0 \end{cases}$$

for $r > R_0$, ~~[illegible]~~ it can be shown

$$\left.\begin{aligned}\left(\frac{\partial}{\partial t} + V_0 \frac{\partial}{\partial r}\right)(r^2 B_r) &= 0\\ \left(\frac{\partial}{\partial t} + V_0 \frac{\partial}{\partial r}\right)(r B_\theta) &= 0\\ \left(\frac{\partial}{\partial t} + V_0 \frac{\partial}{\partial r}\right)(r B_\phi) &= 0\end{aligned}\right\} \text{from combining mag. induction eq. \& } \nabla\cdot\underset{\sim}{B}=0$$

Use method of characteristics to solve

$$B_r(r, \theta, \phi, t) = B_r(r_0, \theta, \phi, t_0)$$

$$B_{\theta,\phi}(r, \theta, \phi, t) = B_{\theta,\phi}(r_0, \theta, \phi, t_0)$$

r_0, t_0 are related to r, t via the characteristic eq. which is from

$$\frac{dr}{dt} = V_0 \Rightarrow r - r_0 = V_0(t - t_0)$$

$$\boxed{t_0 = t - \frac{r - r_0}{V_0}}$$

also we apply B.C.'s at the Sun as we did before

$$\boxed{\frac{B_\theta}{B_r} = \frac{V_\theta}{V_0} \; ; \quad \frac{B_\phi}{B_r} = \frac{V_\phi}{V_0} = \frac{R_0 \Omega_0 \sin\theta + V_\phi'}{V_0}}$$

General Solution

$$B_r = B_0 \left(\frac{R_0}{r}\right)^2$$

$$B_\theta = - B_0 \left(\frac{R_0}{r}\right) \frac{V_\theta(\theta, \phi, t_0)}{V_0}$$

$$B_\phi = - B_0 \left(\frac{R_0}{r}\right) \frac{R_0 \Omega_0 \sin\theta + V_\phi'(\theta, \phi, t_0)}{V_0}$$

$$\boxed{t_0 = t - \frac{r - R_0}{V_0}}$$