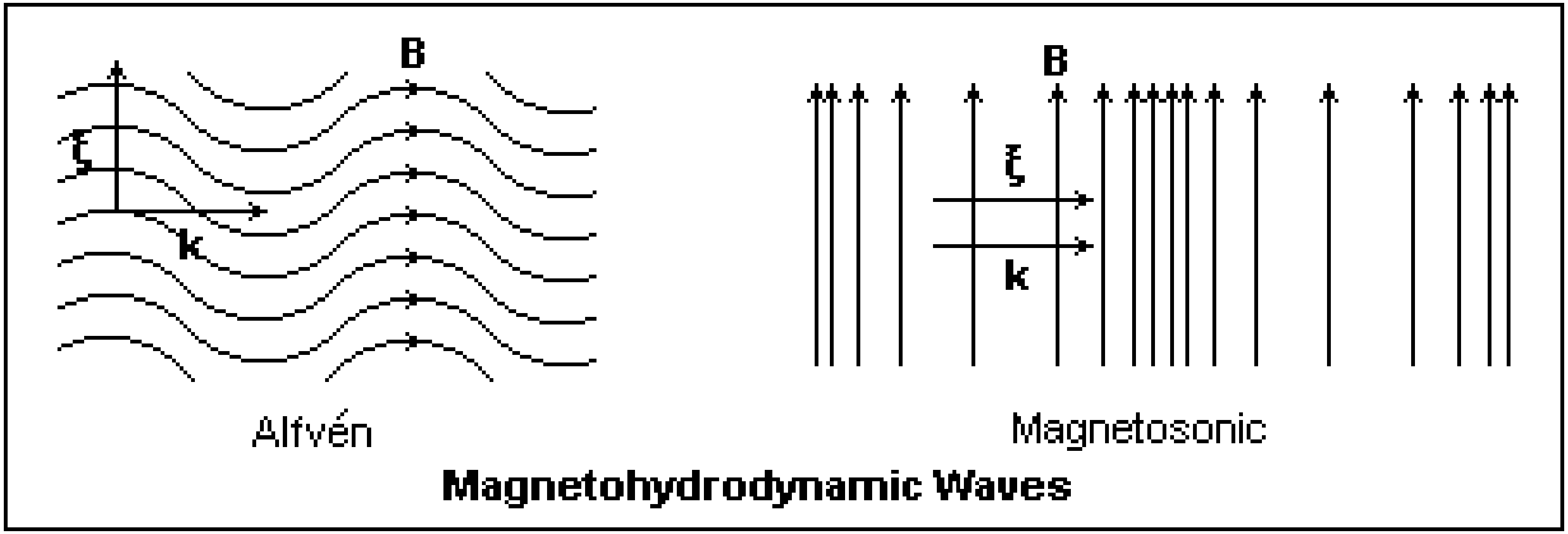
PTYS558 – March 30, 2020

MHD Plasma Waves (continued)

Recall that we previously discussed an intuitive picture for magnetic waves in an ideal MHD fluid



Today we will quantify this.

Last Wednesday (March 25 lecture), we derived the general dispersion relation for (ideal) MHD waves including magnetic stresses and the gravitational force:

$$\boxed{\begin{aligned}\omega^2\vec{u}' \;=\; & C_s^2\,\vec{k}\left(\vec{k}\cdot\vec{u}'\right) \;-\; ik(\gamma-1)\vec{g}\left(\vec{k}\cdot\vec{u}'\right) \\ & +\; ig\vec{k}u_r' + \frac{1}{4\pi\rho_0}\left(\vec{k}\times\vec{k}\times\left(\vec{u}'\times\vec{B}_0\right)\right)\times\vec{B}_0\end{aligned}}$$

To quantify this intuitive picture, return to general dispersion relation, with $\underset{\sim}{g} = 0$, $\varsigma = 0$ (mag. waves only)

$$\omega^2 \underset{\sim}{u}' = \left\{ \underset{\sim}{k} \times \left[\underset{\sim}{k} \times \left(\underset{\sim}{u}' \times \underset{\sim}{B}_0 \right) \right] \right\} \times \frac{\underset{\sim}{B}_0}{4\pi \rho_0}$$

$$= c_A^2 \left[\underset{\sim}{k} \times \left\{ \underset{\sim}{k} \times \left(\underset{\sim}{u}' \times \underset{\sim}{b}_0 \right) \right\} \right] \times \underset{\sim}{b}_0$$

$$\underset{\sim}{b}_0 = \underset{\sim}{B}_0 / B_0$$

$$= c_A^2 \left[\underset{\sim}{k} \times \left\{ \underset{\sim}{u}' \left(\underset{\sim}{k} \cdot \underset{\sim}{b}_0 \right) - \underset{\sim}{b}_0 \left(\underset{\sim}{k} \cdot \underset{\sim}{u}' \right) \right\} \right] \times \underset{\sim}{b}_0$$

$$\vdots$$

$$\frac{\omega^2}{c_A^2} \underset{\sim}{u}' = \left(\underset{\sim}{k} \cdot \underset{\sim}{b}_0 \right)^2 \underset{\sim}{u}' - \left(\underset{\sim}{k} \cdot \underset{\sim}{u}' \right) \left(\underset{\sim}{b}_0 \cdot \underset{\sim}{k} \right) \underset{\sim}{b}_0 + \left[\underset{\sim}{k} \cdot \underset{\sim}{u}' - \left(\underset{\sim}{k} \cdot \underset{\sim}{b}_0 \right) \left(\underset{\sim}{b}_0 \cdot \underset{\sim}{u}' \right) \right] \underset{\sim}{k}$$

define $\underset{\sim}{k} \cdot \underset{\sim}{b}_0 = k \cos\theta$ $\qquad$ θ = angle between $\underset{\sim}{k}$ & $\underset{\sim}{B}_0$

$$\boxed{\begin{aligned}\frac{\omega^2}{c_A^2}\underset{\sim}{u}' = k^2\cos^2\theta\, \underset{\sim}{u}' - (\underset{\sim}{k}\cdot\underset{\sim}{u}')\, k\cos\theta\, \underset{\sim}{b}_0 \\ + \left[\underset{\sim}{k}\cdot\underset{\sim}{u}' - k\cos\theta\,(\underset{\sim}{b}_0\cdot\underset{\sim}{u}')\right]\underset{\sim}{k}\end{aligned}} \quad (1)$$

dot into $\underset{\sim}{b}_0$ $\Rightarrow$ $\underset{\sim}{u}'\cdot\underset{\sim}{b}_0 = 0$

$\Rightarrow$ velocity perturbation is normal to $\underset{\sim}{B}$.

$\underset{\sim}{J}\times\underset{\sim}{B}$ is normal to $\underset{\sim}{B}$; not surprising

Dot into $\underset{\sim}{k}$ to give

$$\frac{\omega^2}{c_A^2}\underset{\sim}{u}'\cdot\underset{\sim}{k} = (\underset{\sim}{k}\cdot\underset{\sim}{u}')\, k^2$$

$$\Rightarrow \quad \left(\frac{\omega^2}{c_A^2} - k^2\right)(\underset{\sim}{k}\cdot\underset{\sim}{u}') = 0$$

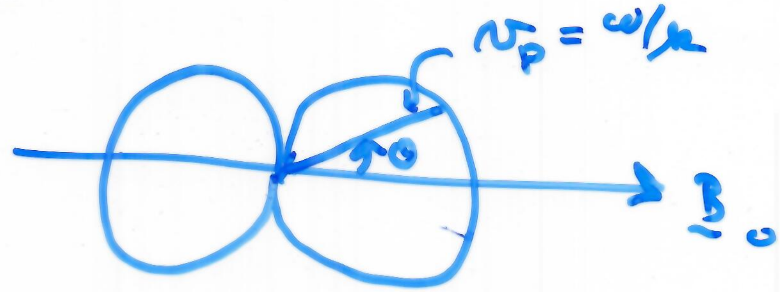
<u>2 cases</u>:

(i) SHEAR Alfven wave

$$\underset{\sim}{k}\cdot\underset{\sim}{u}' = 0 \Rightarrow \frac{\omega^2}{c_A^2} = k^2\cos^2\theta$$

$$\omega = \pm k c_A \cos\theta$$

"polar diagram"



Shear Alfven wave

turns out that this wave has NO density or pressure variation!

can show $\frac{\partial \rho'}{\partial t} = 0$ $\frac{\partial p'}{\partial t} = 0$ for uniform ρ_0, p_0

Can also be shown (from linearized mag. induction eq. assuming plane wave soln)

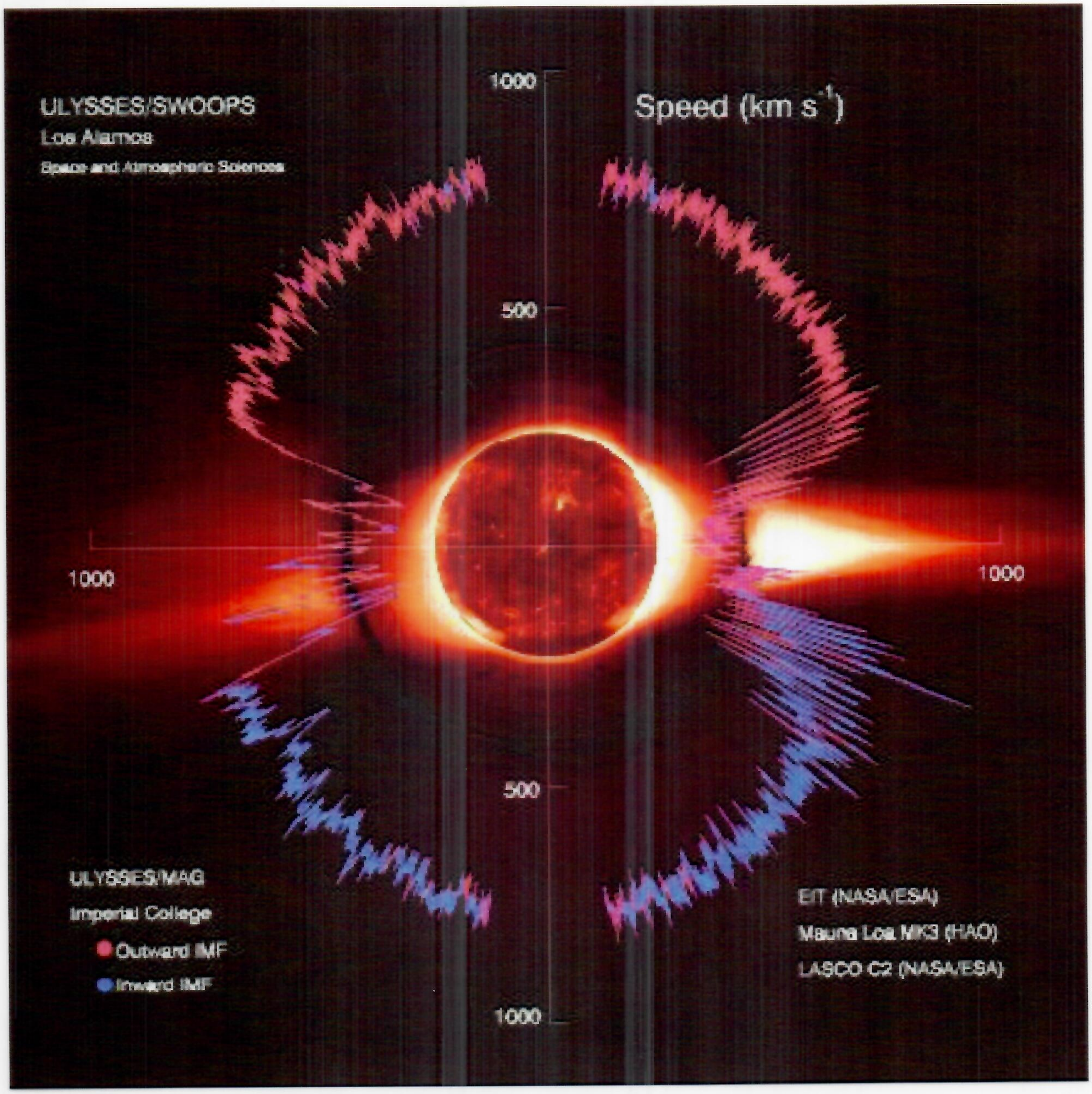
$$\underset{\sim}{u}' = -c_A \underset{\sim}{b}' = -c_A \underset{\sim}{B}'/B_0$$

velocity perturbation is in phase with $\underset{\sim}{B}$.

you can also demonstrate that this mode is caused by the tension part of $\underset{\sim}{J} \times \underset{\sim}{B}$ force.

Also, it can be shown

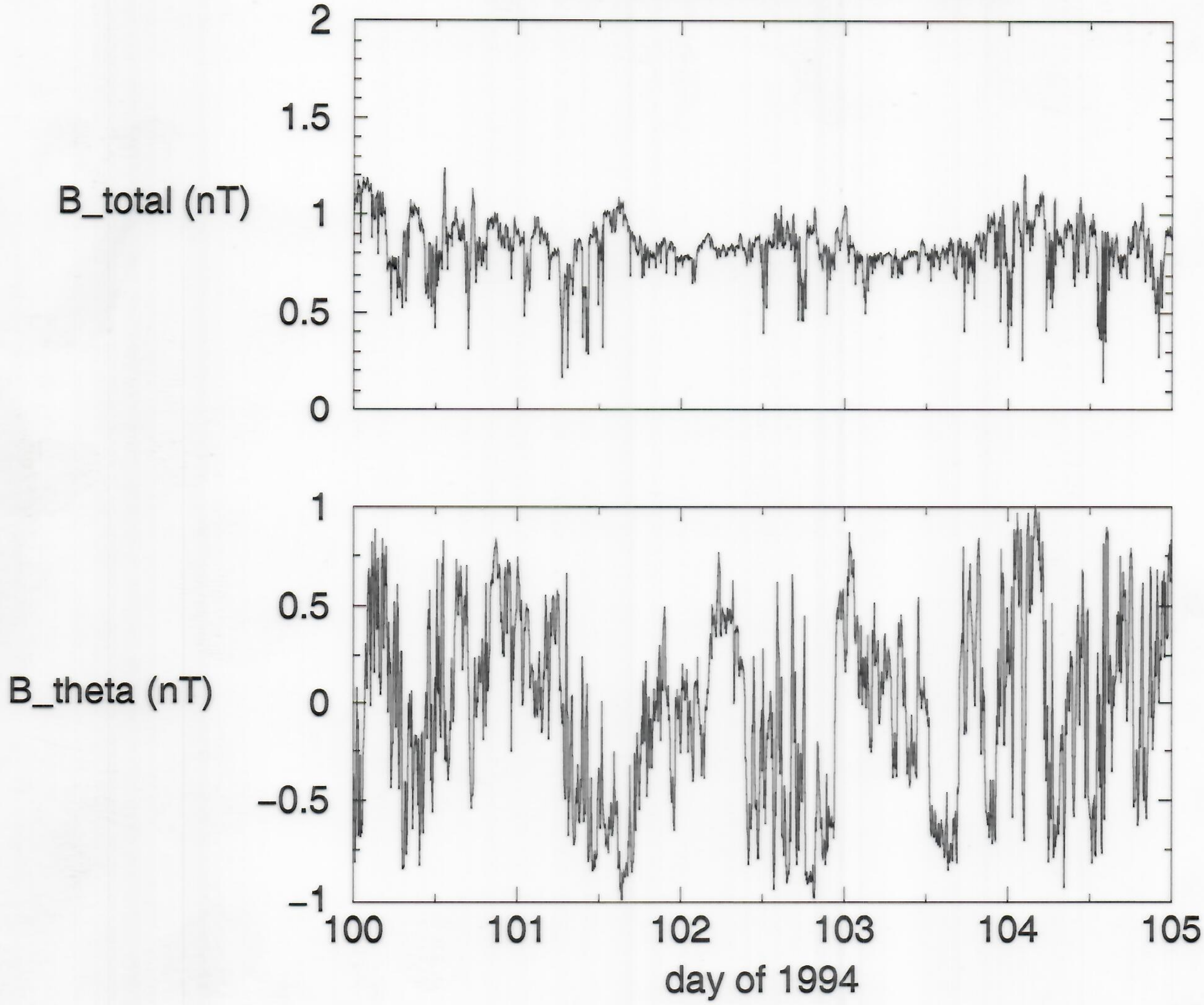
$$\left.\begin{aligned} |\underset{\sim}{B}_0 + \underset{\sim}{B}'| &= \text{constant.} \\ \underset{\sim}{u}' &= -c_A \underset{\sim}{b}' \end{aligned}\right\} \text{exact solutions to original eqs}$$

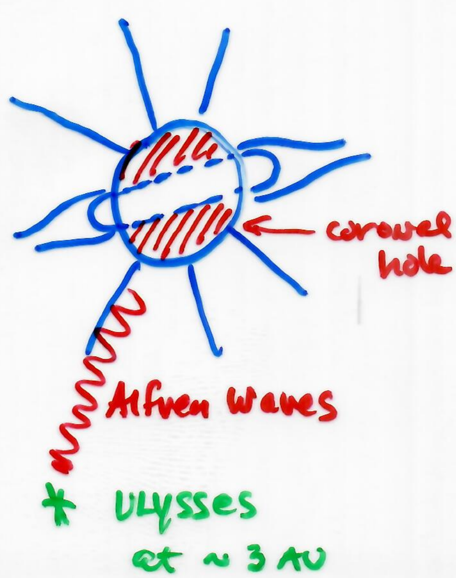


ULYSSES/SWOOPS
Los Alamos
Space and Atmospheric Sciences
Speed (km s^{-1})
1000
500
1000
1000
500
1000
ULYSSES/MAG
Imperial College
Outward IMF
Inward IMF
EIT (NASA/ESA)
Mauna Loa MK3 (HAO)
LASCO C2 (NASA/ESA)

Ulysses Magnetic Field Data

r = 3.3 AU, latitude = −60° South





the shear Alfven wave never decays (in ideal, non-~~dissipative~~ resistive MHD)

Since plasmas are highly conducting these waves take a long time to decay.

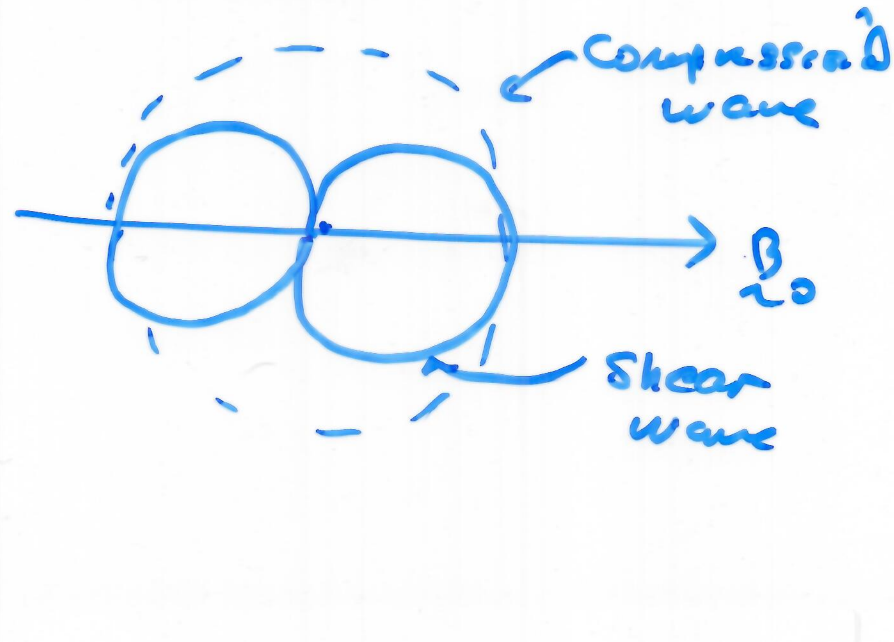
→ these waves, produced near the Sun, are always seen @ 1 AU and beyond

The other mode, case 2

<u>Compressional Alfven wave</u> $\underline{k} \cdot \underline{u}' \neq 0$

$$\Rightarrow \quad \omega^2 = k^2 C_A^2 \Rightarrow \omega = \pm k C_A$$

"like" a sound wave



Magneto-sonic waves

take $\underset{\sim}{g} = 0$

$$\omega^2 \underset{\sim}{u}' = c_s^2 \underset{\sim}{k} (\underset{\sim}{k} \cdot \underset{\sim}{u}') + \frac{1}{4\pi\rho_0} \{ \underset{\sim}{k} \times [\underset{\sim}{k} \times (\underset{\sim}{u}' \times \underset{\sim}{B}_0)] \} \times \underset{\sim}{B}_0$$

$$\vdots$$

$$\frac{\omega^2}{c_A^2} \underset{\sim}{u}' = k^2 \cos^2\theta \, \underset{\sim}{u}' - (\underset{\sim}{k} \cdot \underset{\sim}{u}') k \cos\theta \, \underset{\sim}{b}_0$$

$$+ \left\{ \left[1 + \left(\frac{c_s}{c_A} \right)^2 \right] (\underset{\sim}{k} \cdot \underset{\sim}{u}') - k \cos\theta \, (\underset{\sim}{b}_0 \cdot \underset{\sim}{u}') \right\} \underset{\sim}{k}$$

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Magneto-sonic waves

take $\mathbf{g} = 0$

$$\omega^2 \mathbf{u}' = c_s^2 \mathbf{k} (\mathbf{k} \cdot \mathbf{u}') + \frac{1}{4\pi\rho_0} \{\mathbf{k} \times [\mathbf{k} \times (\mathbf{u}' \times \mathbf{B}_0)]\} \times \mathbf{B}_0$$

$$\vdots$$

$$\frac{\omega^2}{c_A^2} \mathbf{u}' = k^2 \cos^2\theta \, \mathbf{u}' - (\mathbf{k} \cdot \mathbf{u}') k \cos\theta \, \mathbf{b}_0 + \left\{ \left[1 + \left(\frac{c_s}{c_A}\right)^2\right] (\mathbf{k} \cdot \mathbf{u}') - k \cos\theta (\mathbf{b}_0 \cdot \mathbf{u}') \right\} \mathbf{k} \qquad (1)$$

dot ~~this~~ (1) into $\mathbf{k}$

$$\frac{\mathbf{u}' \cdot \mathbf{k}}{\mathbf{u}' \cdot \mathbf{b}_0} = \frac{-k^3 \cos\theta}{\frac{\omega^2}{c_A^2} - \left[1 + \left(\frac{c_s}{c_A}\right)^2\right] k^2} \qquad (2)$$

dot (1) ~~with~~ into $\mathbf{b}_0$ to give

$$\frac{\mathbf{u}' \cdot \mathbf{k}}{\mathbf{u}' \cdot \mathbf{b}_0} = -\frac{\omega^2}{c_s^2 k \cos\theta} \qquad (3)$$

equate (2) & (3)

$$\frac{-k^3\cos\theta}{\frac{\omega^2}{c_A^2} - \left[1+\left(\frac{c_s}{c_A}\right)^2\right]k^2} = -\frac{\omega^2}{c_s^2 k\cos\theta}$$

leads to

$$\boxed{\omega^4 - \omega^2 k^2\left(c_A^2 + c_s^2\right) + k^4 c_s^2 c_A^2 \cos^2\theta = 0}$$

general dispersion relation for magneto-acoustic waves

solution

$$\omega^2 = \frac{k^2\left(c_A^2+c_s^2\right) \pm \sqrt{k^4\left(c_A^2+c_s^2\right)^2 - 4k^4 c_s^2 c_A^2\cos^2\theta}}{2}$$

$$\vdots$$

$$\boxed{v_{phase} = \frac{\omega}{k} = \pm\left\{\frac{1}{2}\left(c_A^2+c_s^2\right) \pm \frac{1}{2}\left[c_A^4 + c_s^4 - 2c_A^2 c_s^2 \cos 2\theta\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}}$$

general phase speed of MHD waves

fast mode $\rightarrow$ $\left(c_A^2 + c_s^2\right)^{1/2}$

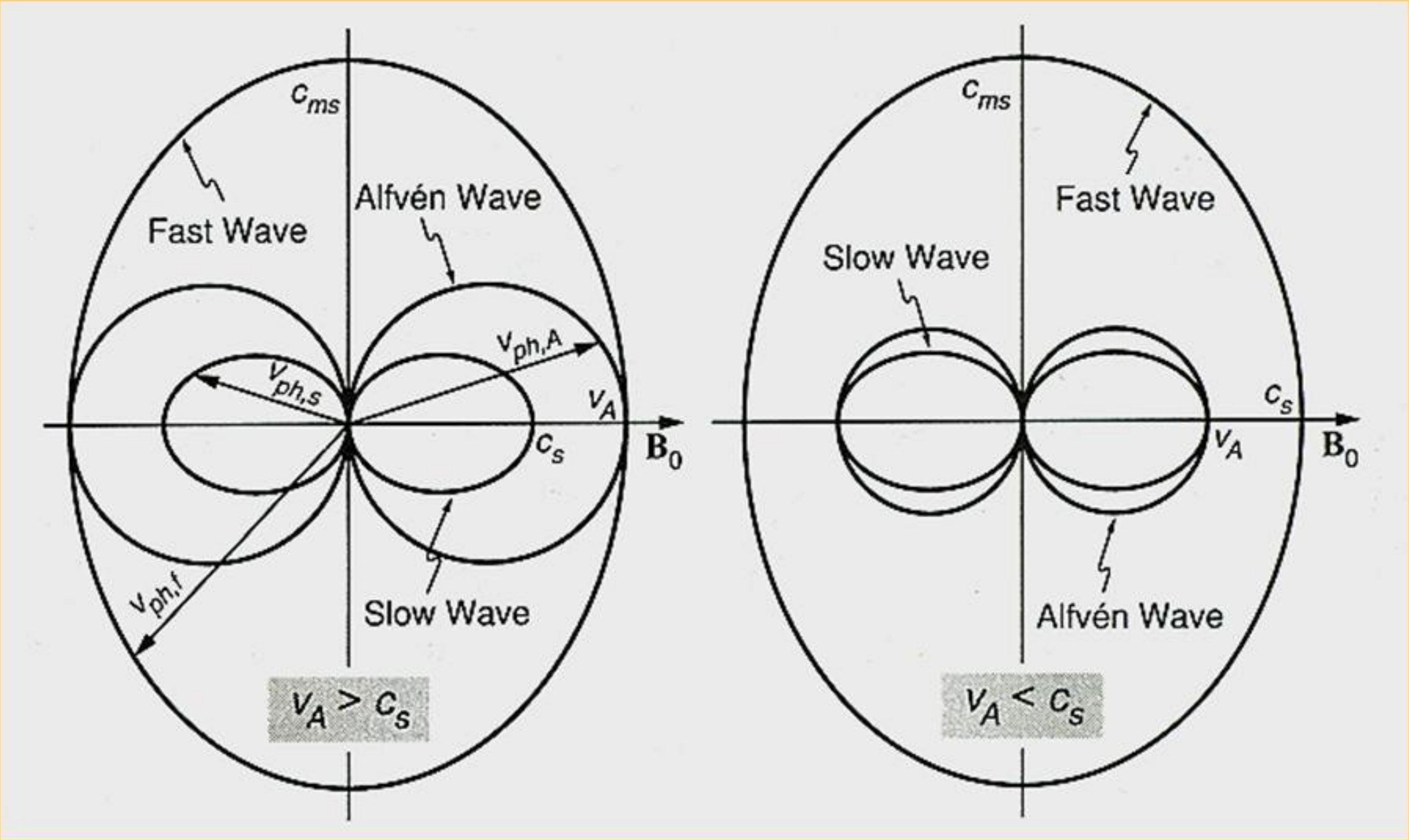
intermediate mode $\rightarrow$ $c_A\cos\theta$

slow mode $\rightarrow$ not easily written here.

Dispersion relation for magneto-sonic waves in an ideal MHD fluid

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(C_A^2 + C_S^2) \pm (C_A^4 + C_S^4 - 2C_A^2 C_S^2 \cos 2\theta)^{1/2}$$

Phase-velocity polar diagram of MHD waves



Decay of Shear Alfven waves

approximate determination of the scale over which the shear Alfven wave decays — NOT damps!

what is effect of resistivity?

look at MHD energy eq.

$$\frac{\partial}{\partial t}(\text{Energy density}) + \nabla \cdot (\text{Energy flux}) = \underset{\sim}{j} \cdot \underset{\sim}{E}$$

ignore pressure, drop 2nd-order terms

$$\rightarrow \quad \underset{\sim}{j} \cdot \underset{\sim}{E} = 0$$

$$\underset{\sim}{j} = \frac{c}{4\pi} \nabla \times \underset{\sim}{B} \qquad \underset{\sim}{E} = -\frac{1}{c} \underset{\sim}{u} \times \underset{\sim}{B} + \eta \underset{\sim}{J}$$

$$\underset{\sim}{j} \cdot \underset{\sim}{E} = \underset{\sim}{j} \cdot \left[-\frac{1}{c} \underset{\sim}{u} \times \underset{\sim}{B} + \eta \underset{\sim}{J} \right]$$

$$= \underbrace{-\frac{1}{c} \cdot \frac{c}{4\pi} (\nabla \times \underset{\sim}{B}) \cdot (\underset{\sim}{u} \times \underset{\sim}{B})}_{\text{can be manipulated to give}} + \eta J^2$$

$$\nabla \cdot \left(\frac{c}{4\pi} \underset{\sim}{E}^* \times \underset{\sim}{B} \right)$$

$$\underset{\sim}{E}^* = -\frac{1}{c} \underset{\sim}{u} \times \underset{\sim}{B}$$

$$\nabla\cdot\mathbf{S} = -Q \qquad Q = \eta J^2$$

$$-\frac{1}{c}\mathbf{u}\times\mathbf{B}$$

$$\mathbf{S} = \left(\frac{c}{4\pi}\mathbf{E}^*\times\mathbf{B}\right)$$

evaluate $\mathbf{S}$ + Q

really want $\langle\mathbf{S}\rangle$ → $\langle\ \rangle$ = avg. over wave period | wavelength

$\langle Q\rangle$

$$\langle\mathbf{S}\rangle = \left\langle\frac{c}{4\pi}\mathbf{E}^*\times\mathbf{B}\right\rangle$$

$$= \frac{c}{4\pi}\left\langle\left(-\frac{1}{c}\mathbf{u}\times\mathbf{B}\right)\times\mathbf{B}\right\rangle$$

$$\sim -\frac{1}{4\pi}\langle\delta\mathbf{u}\cdot\delta\mathbf{B}\rangle\,\mathbf{B}_0$$

$-c_A\,\delta\mathbf{B}/B_0$ for shear Alfven wave

$$\langle\mathbf{S}\rangle \simeq \frac{c_A}{4\pi}\langle\underbrace{\delta B^2}_{(B')^2}\rangle\,\hat{z} \longleftarrow$$

propagation direction
direction of $\mathbf{B}_0$
(take $\mathbf{B}_0 = B_0\hat{z}$)

$$\langle Q\rangle = \langle\eta J^2\rangle = \eta\frac{c^2}{(4\pi)^2}\langle(\nabla\times\mathbf{B})^2\rangle$$

$$= \eta\frac{c^2}{(4\pi)^2}k^2\langle\underbrace{\delta B^2}_{(B')^2}\rangle$$

$\mathbf{J}' = \frac{c}{4\pi}\nabla\times\mathbf{B}'$

$\mathbf{J}'(\mathbf{k},\omega) = \frac{c}{4\pi}(-i)\mathbf{k}\times\mathbf{B}'$

$\mathbf{J}'\mathbf{J}'^* = J^2 = \left(\frac{c}{4\pi}\right)^2(\mathbf{k}\times\mathbf{B}')^2$

this gives

$$\nabla\cdot\underset{\sim}{S} \rightarrow \frac{d}{dz}\langle \underset{\sim}{S}\rangle_z = \frac{d}{dz}\left(\frac{c_A}{4\pi}\langle \delta B^2\rangle\right)$$

$$= -\langle Q\rangle = -\eta\frac{c^2}{(4\pi)^2}k^2\langle \delta B^2\rangle$$

$$\therefore \quad \frac{d}{dz}\langle \delta B^2\rangle = -\frac{\eta c^2 k^2}{c_A}\langle \delta B^2\rangle$$

$$\Rightarrow \quad \langle \delta B^2\rangle \sim e^{-\frac{\eta c^2 k^2}{4\pi c_A} z}$$

η is very small, $\Rightarrow$ length scale of decay

$$l_d \sim \frac{4\pi c_A}{\eta c^2 k^2} \text{ is very large}$$

this approach works for waves of wavelength much smaller than l_d.

Plasma oscillations

Consider ~~a perturbation of~~ the electron momentum eq.

$$m_e n_e \frac{\partial \underset{\sim}{u}_e}{\partial t} + m_e n_e \underset{\sim}{u}_e \cdot \nabla \underset{\sim}{u}_e = -\nabla P_e - e n_e \underset{\sim}{E}$$

$$- \frac{e}{c} n_e \underset{\sim}{u}_e \times \underset{\sim}{B}$$

~~take~~ no magnetic field (un-magnetized), no pressure (cold)

$\rightarrow$ cold, unmagnetized plasma

$$m_e n_e \frac{\partial \underset{\sim}{u}_e}{\partial t} + m_e n_e \underset{\sim}{u}_e \cdot \nabla \underset{\sim}{u}_e = -e n_e \underset{\sim}{E}$$

$$\nabla \cdot \underset{\sim}{E} = 4\pi e (n_p - n_e)$$

$$\frac{\partial n_e}{\partial t} + n_e \nabla \cdot \underset{\sim}{u}_e + \underset{\sim}{u}_e \cdot \nabla n_e = 0$$

$\underset{\sim}{u}_e = \underset{\sim}{u}'$ (no initial flow)

$\underset{\sim}{E} = \underset{\sim}{E}'$

$n_e = n_0 + n'$ $(n_p = n_0)$

$$m_e n_0 \frac{\partial \underset{\sim}{u}'}{\partial t} = -e n_0 \underset{\sim}{E}'$$

$$\nabla \cdot \underset{\sim}{E}' = -4\pi e n'$$

$$\frac{\partial n'}{\partial t} = -n_0 \nabla \cdot \underset{\sim}{u}'$$

$\nabla \cdot$ (top eq.), subst. from bottom eq. & middle eq.

$$m_e n_0 \frac{\partial}{\partial t}\left(-\frac{1}{n_0}\frac{\partial n'}{\partial t}\right) = -e n_0 (-4\pi e n')$$

$$\frac{\partial^2 n'}{\partial t^2} = \frac{4\pi n_0 e^2}{m_e} n'$$

$$\Rightarrow \quad n' \sim e^{i\omega t} \qquad \underbrace{\omega = \left(\frac{4\pi n_0 e^2}{m_e}\right)^{1/2}}_{\omega_e}$$

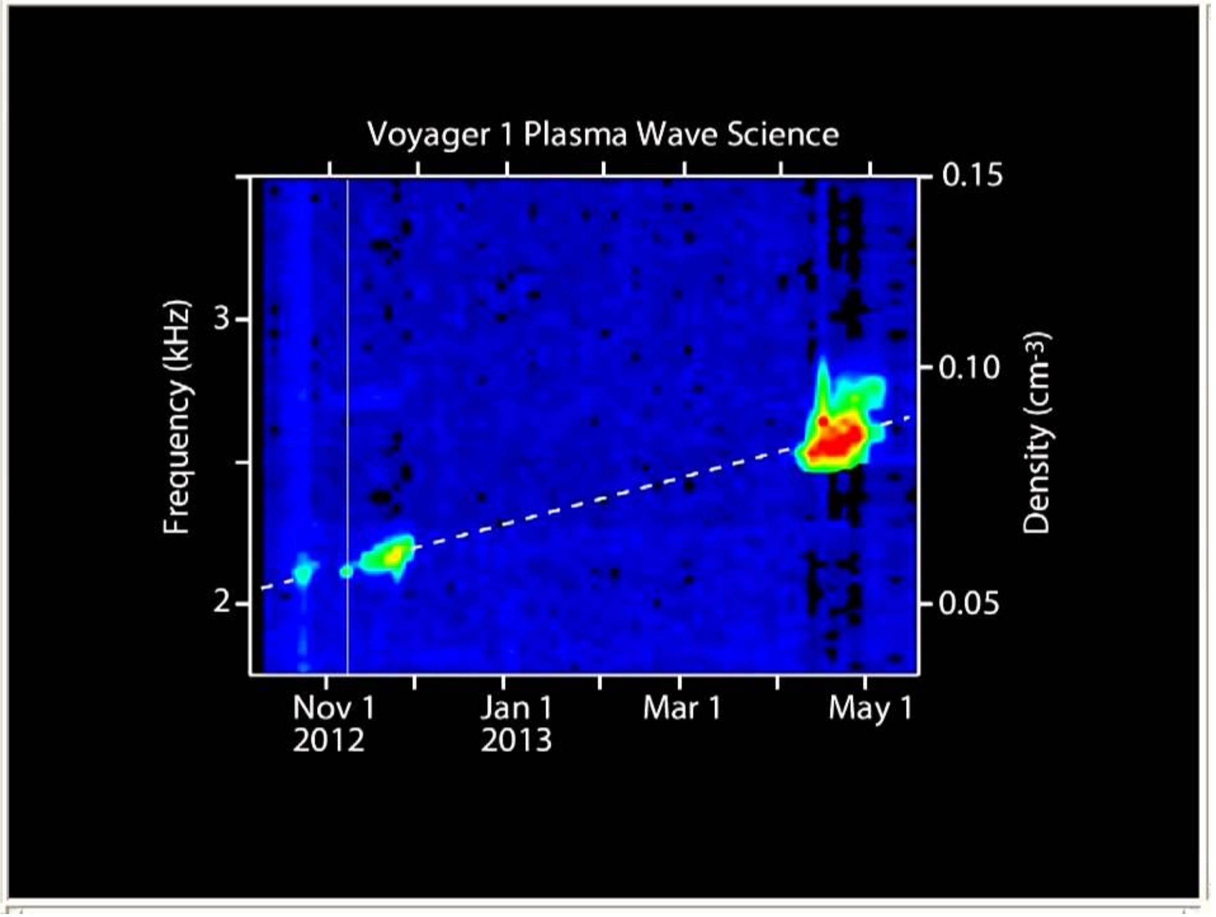
electron plasma freq.

if we include the pressure term, but no mag. field, we get

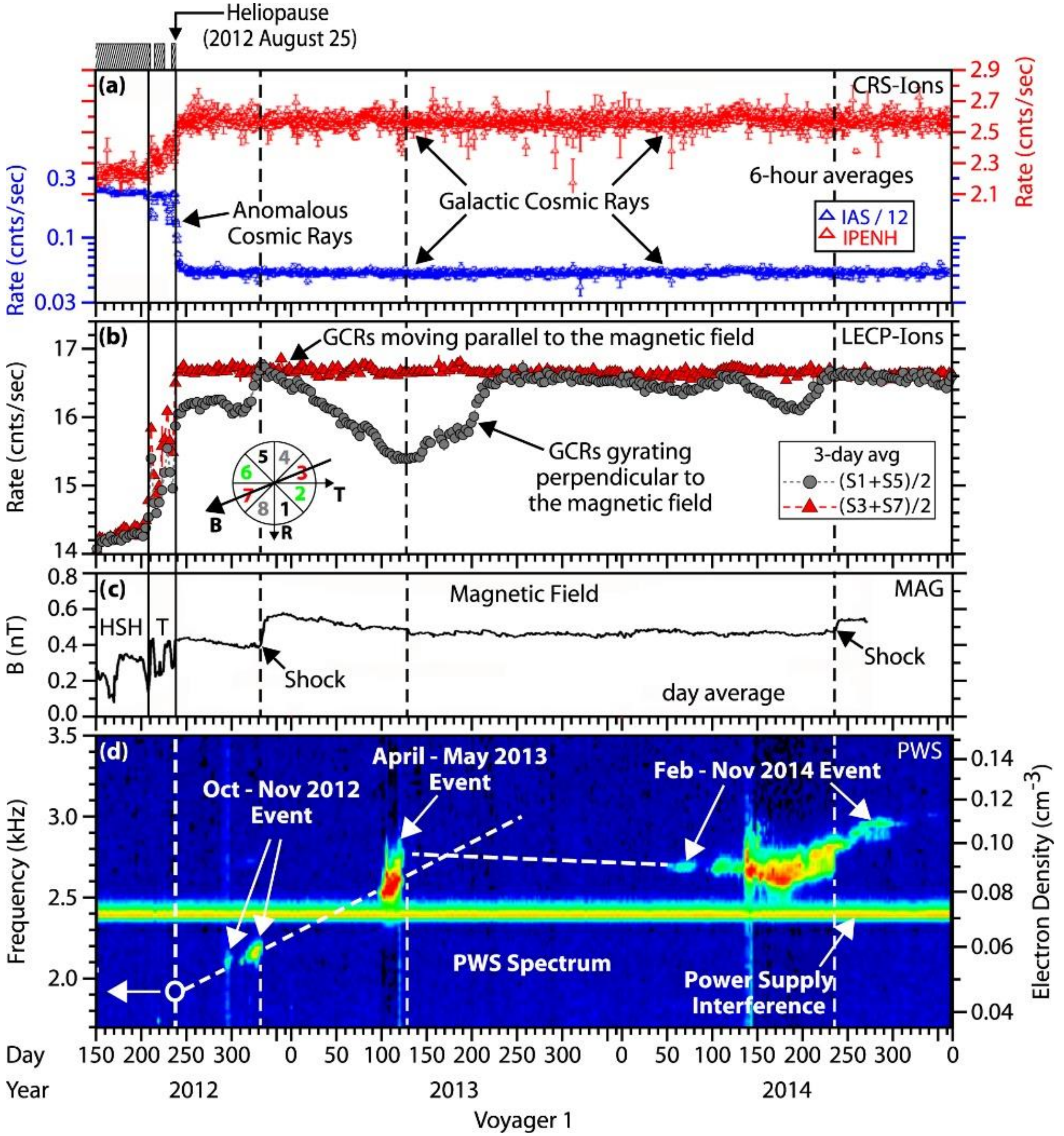
$$\boxed{\omega^2 = \omega_e^2 + c_s^2 k^2} \qquad c_s = \frac{\gamma P_e}{\rho_e} \quad \text{adiabatic electrons}$$

plasma oscillations in warm plasma

Plasma waves detected by Voyager in distant heliosphere (Voyager at ~120 AU)



These were interpreted as Langmuir waves, with a frequency equal to the electron plasma frequency, generated locally at Voyager 1’s location, by a disturbance, presumably a shock wave


Heliopause
(2012 August 25)
(a)
CRS-Ions
Rate (cnts/sec)
Rate (cnts/sec)
Anomalous
Cosmic Rays
Galactic Cosmic Rays
6-hour averages
IAS / 12
IPENH
(b)
GCRs moving parallel to the magnetic field
LECP-Ions
GCRs gyrating
perpendicular to
the magnetic field
3-day avg
(S1+S5)/2
(S3+S7)/2
(c)
Magnetic Field
MAG
HSH
T
B (nT)
Shock
Shock
day average
(d)
PWS
April - May 2013
Event
Feb - Nov 2014 Event
Oct - Nov 2012
Event
Frequency (kHz)
Electron Density (cm^{-3})
PWS Spectrum
Power Supply
Interference
Day
Year
2012
2013
2014
Voyager 1