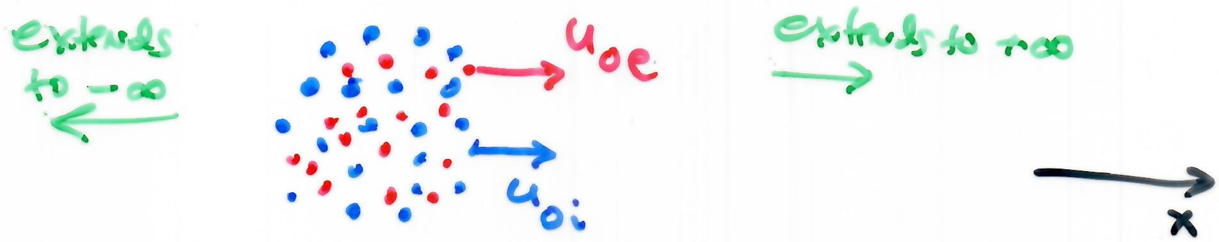
PTYS558: April 8, 2020

- Two-stream instability in a proton-electron plasma
- Turbulence

2-stream instability in a plasma



this is a 1-D problem only variations in x direction.

for simplicity, assume cold plasma, un-magnetized

Basic equations

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0$$

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e u_e) = 0$$

$$n_i m_i \frac{\partial u_i}{\partial t} + n_i m_i u_i \frac{\partial u_i}{\partial x} = e n_i E$$

$$n_e m_e \frac{\partial u_e}{\partial t} + n_e m_e u_e \frac{\partial u_e}{\partial x} = -e n_e E$$

$$\frac{dE}{dx} = 4\pi e(n_i - n_e)$$

perturb the plasma

$$n_i = n_0 + n_i' \qquad n_i' \ll n_0$$
$$n_e = n_0 + n_e' \qquad n_e' \ll n_0$$
$$u_i = u_{0i} + u_i'$$
$$u_e = u_{0e} + u_e'$$
$$E = E'$$

The linearized eq.'s for perturbed quantities (dropping 2nd order terms)

$$\frac{\partial n_i'}{\partial t} + n_0 \frac{\partial u_i'}{\partial x} + u_{0i} \frac{\partial n_i'}{\partial x} = 0$$

$$\frac{\partial n_e'}{\partial t} + n_0 \frac{\partial u_e'}{\partial x} + u_{0e} \frac{\partial n_e'}{\partial x} = 0$$

$$n_0 m_i \frac{\partial u_i'}{\partial t} + n_0 m_i u_{0i} \frac{\partial u_i'}{\partial x} = e n_0 E'$$

$$n_0 m_e \frac{\partial u_e'}{\partial t} + n_0 m_e u_{0e} \frac{\partial u_e'}{\partial x} = -e n_0 E'$$

$$\frac{dE'}{dx} = 4\pi e (n_i' - n_e')$$

assume plane-wave soln to gain

$$-\omega n_i' + n_0 k u_i' + u_{0i} k n_i' = 0$$

$$-\omega n_e' + n_0 k u_e' + u_{0e} k n_e' = 0$$

$$-\omega n_0 m_i u_i' + n_0 m_i u_{0i} k u_i' = -i e n_0 E'$$

$$-\omega n_0 m_e u_e' + n_0 m_e u_{0e} k u_e' = i e n_0 E'$$

$$k E' = -i 4\pi e (n_i' - n_e')$$

write this

$$\begin{pmatrix} -\omega + u_{0i}k & 0 & n_0 k & 0 & 0 \\ 0 & -\omega + u_{0e}k & 0 & n_0 k & 0 \\ 0 & 0 & -\omega + u_{0i}k & 0 & i\frac{e}{m_i} \\ 0 & 0 & 0 & -\omega + u_{0e}k & -i\frac{e}{m_e} \\ i4\pi e & -i4\pi e & 0 & 0 & k \end{pmatrix} \begin{pmatrix} n_i' \\ n_e' \\ u_i' \\ u_e' \\ E \end{pmatrix} = 0$$

$\underset{\sim}{A} \cdot \underset{\sim}{B} = 0$ $\quad \underset{\sim}{A} \to 5\times5$ matrix

$\underset{\sim}{B} \to 5\times1$ matrix

$|\underset{\approx}{A}| = 0$ gives the dispersion relation

we find

$$|\underset{\approx}{A}| = (-\omega + ku_{0i})\Big\{(-\omega + ku_{0e})(-\omega + ku_{0i})(-\omega + ku_{0e})k + n_0 k\big(-(-\omega + ku_{0i})\frac{4\pi e^2}{m_e}\Big\}$$
$$+ n_0 k\Big\{-(-\omega + u_{0e}k)\, i\frac{e}{m_i}\big(-i 4\pi e(-\omega + ku_{0e})\big)\Big\}$$

divide by $(-\omega + u_{0i}k)^2(-\omega + u_{0e}k)^2$

$$\frac{|\underset{\approx}{A}|}{(\ )^2(\ )^2} = k - \omega_e^2 k\frac{1}{(-\omega + ku_{0e})^2} - \omega_i^2 k\frac{1}{(-\omega + ku_{0i})^2} \underset{\text{set}}{=} 0$$

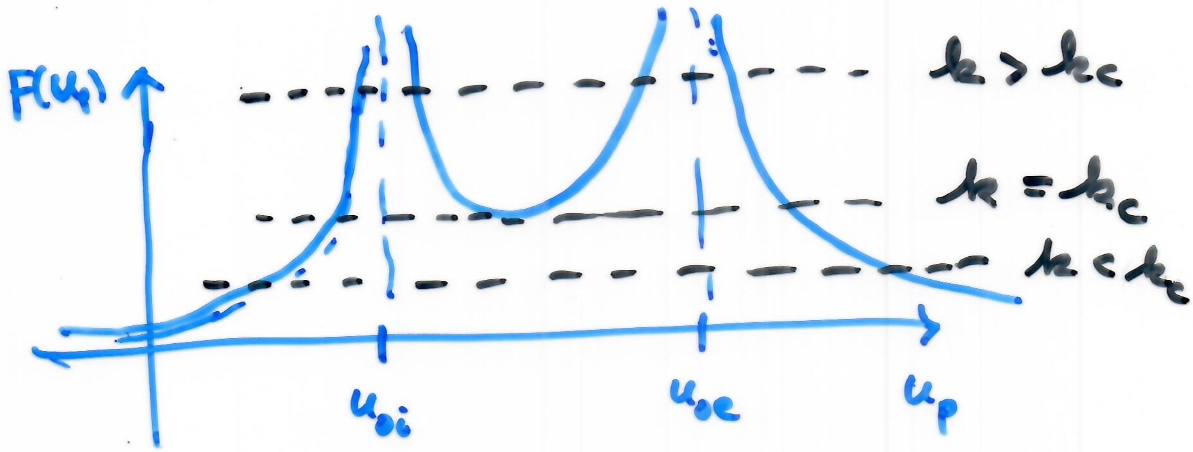
$$1 = \frac{\omega_e^2}{(-\omega + ku_{0e})^2} + \frac{\omega_i^2}{(-\omega + ku_{0i})^2}$$

define ~~[illegible]~~ $u_p = \omega/k$ = phase velocity we get

$$1 = \frac{\omega_e^2}{k^2(-u_p + u_{0e})^2} + \frac{\omega_i^2}{k^2(-u_p + u_{0i})^2}$$

$$\Rightarrow \quad k^2 = \frac{\omega_e^2}{(u_p - u_{0e})^2} + \frac{\omega_i^2}{(u_p - u_{0i})^2} \qquad (*)$$

$$F(u_p) = \frac{\omega_e^2}{(u_p - u_{0e})^2} + \frac{\omega_i^2}{(u_p - u_{0i})^2}$$



$\frac{dF(u_p)}{du_p} \underset{set}{=} 0$, solve for u_p, insert into $(*)$ to get k_c, we find

$$k_c = \frac{\omega_e^{2/3} + \omega_i^{2/3}}{(u_{0e} - u_{0i})^2}$$

$k > k_c \Rightarrow$ 4 real roots $\Rightarrow$ waves.

$k < k_c \Rightarrow$ 2 real roots
2 complex roots

for $k < k_c$, there are 2 complex roots, one of these is associated with wave growth $\Rightarrow$ TWO-STREAM INSTABILITY

Consider a positron - electron plasma and opposing flows $u_{oi} = u_o$

$u_{oe} = -u_o$

$\omega_i = \omega_e$

$$k_c^2 = \sqrt{2}\ \frac{\omega_e}{u_o}$$

and the dispersion relation can be shown to be

$$\omega^2 = k^2 u_o^2 + \omega_e^2 \pm \left(\omega_e^4 + 4k^2\omega_e^2 u_o^2\right)^{1/2}$$

Instability occurs ~~...~~ when $k < k_c$. ~~...~~

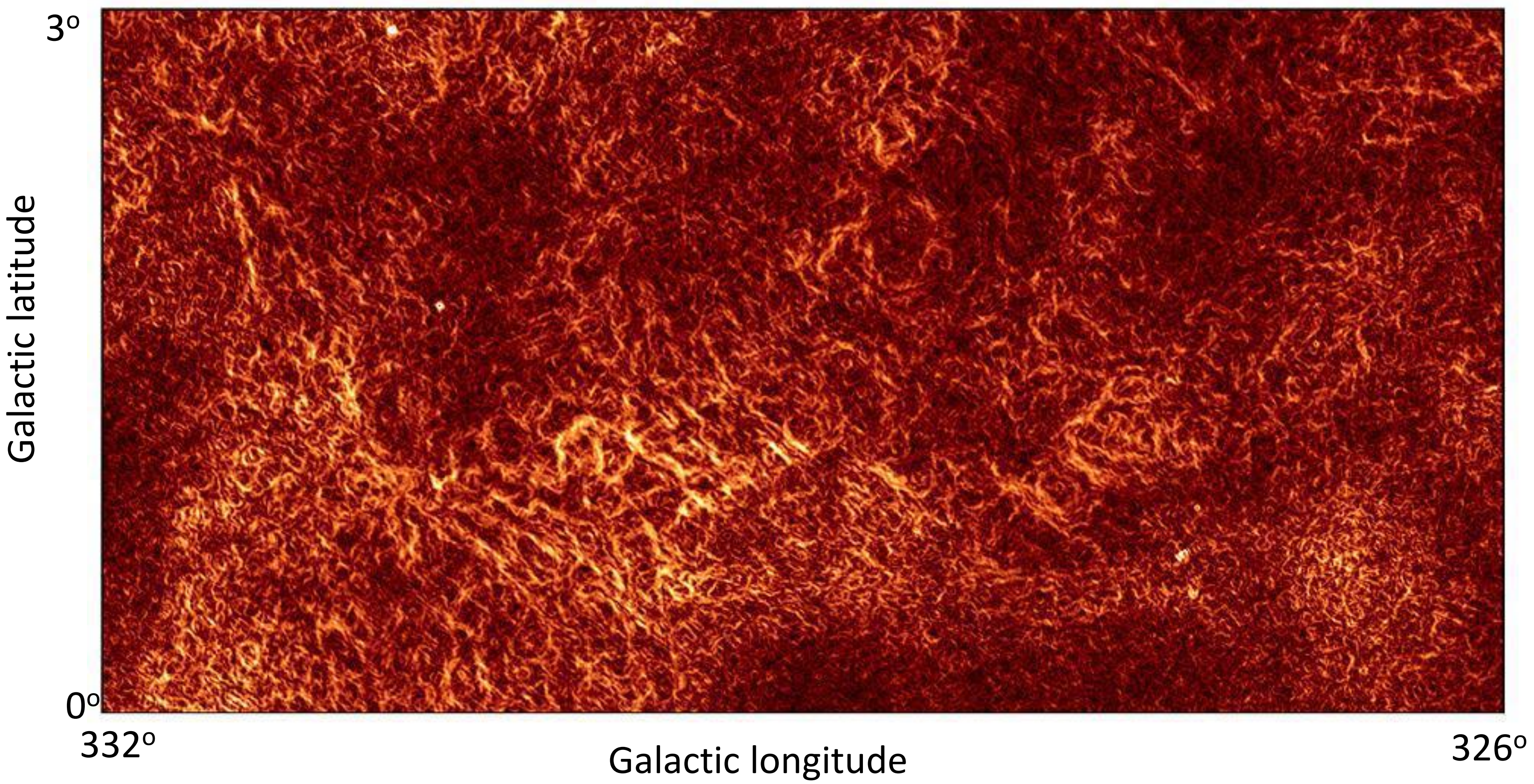
max. growth occurs when $\frac{d\omega}{dk} = 0$, we find this to happen at $k_m = \frac{\sqrt{3}}{2}\frac{\omega_e}{u_o}$ ← k_m at max. growth

$\Rightarrow$ growth rate of $\frac{1}{2}\omega_e$

$\gamma = \frac{1}{2}\omega_e \qquad \gamma \rightarrow$ growth rate

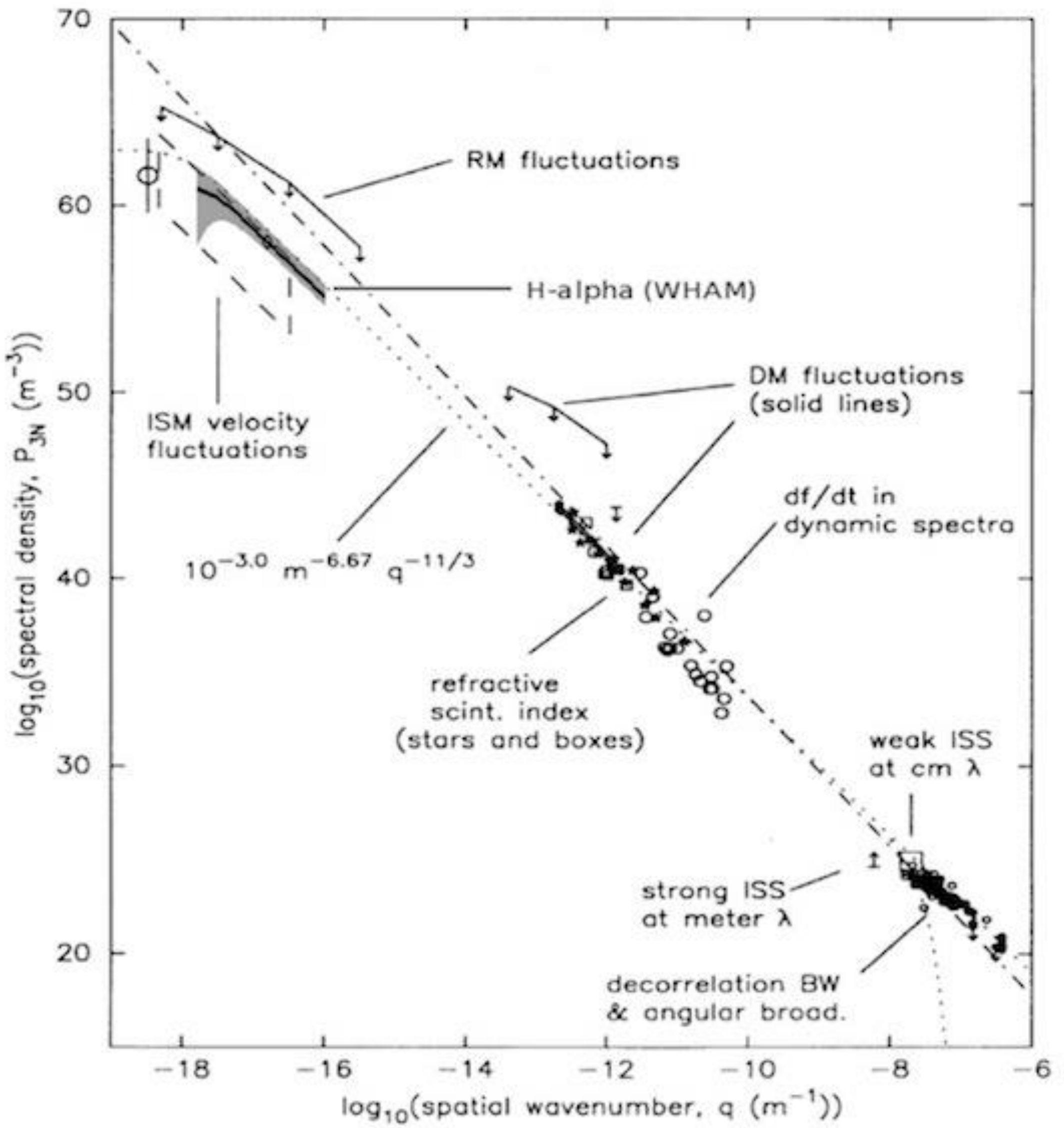
Interstellar Turbulence

18-sq. deg. color-coded map of $|\nabla P|$ (**P** is the 2-component Stokes polarization vector) for 1.4GHz radio observations from ATCA (Australia Telescope Compact Array) radio telescope

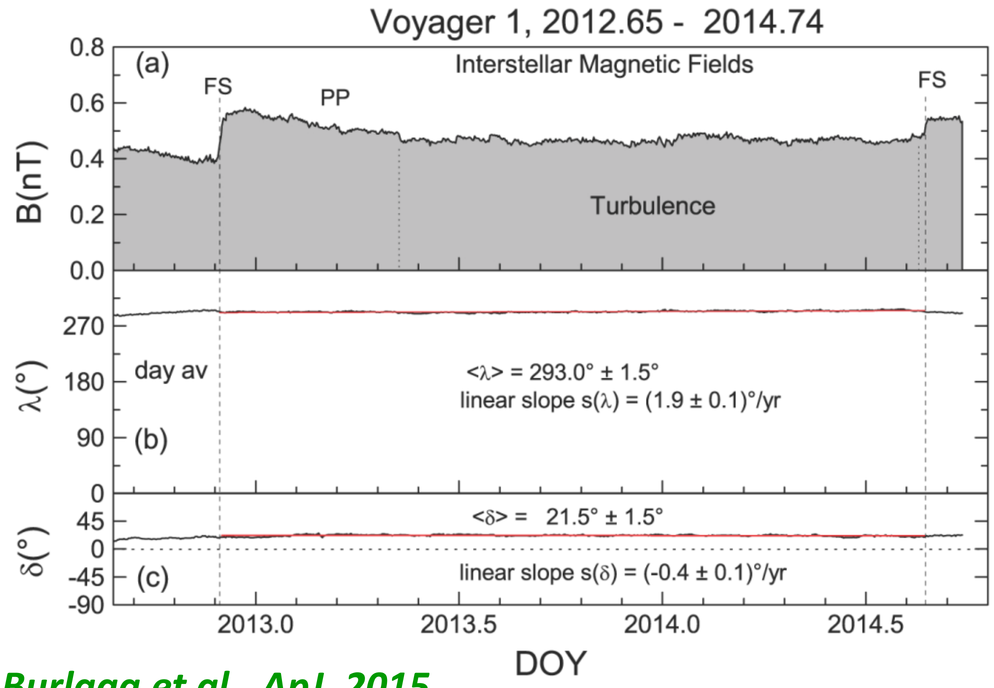


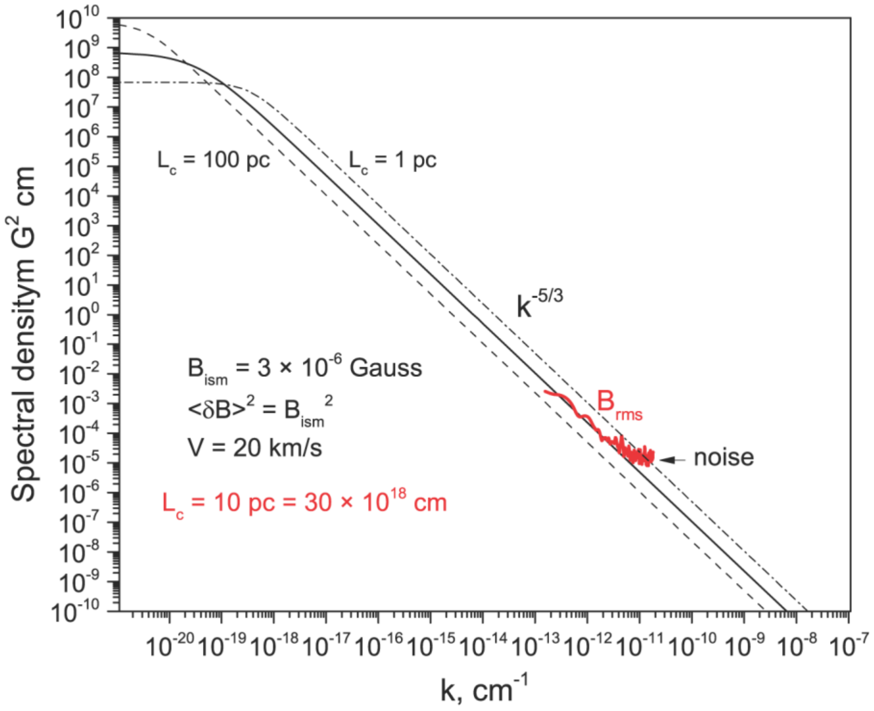
Gaensler et al., Nature, 2011

The "Great Power Law in the Sky"
(S. Spangler, Univ. of Iowa)



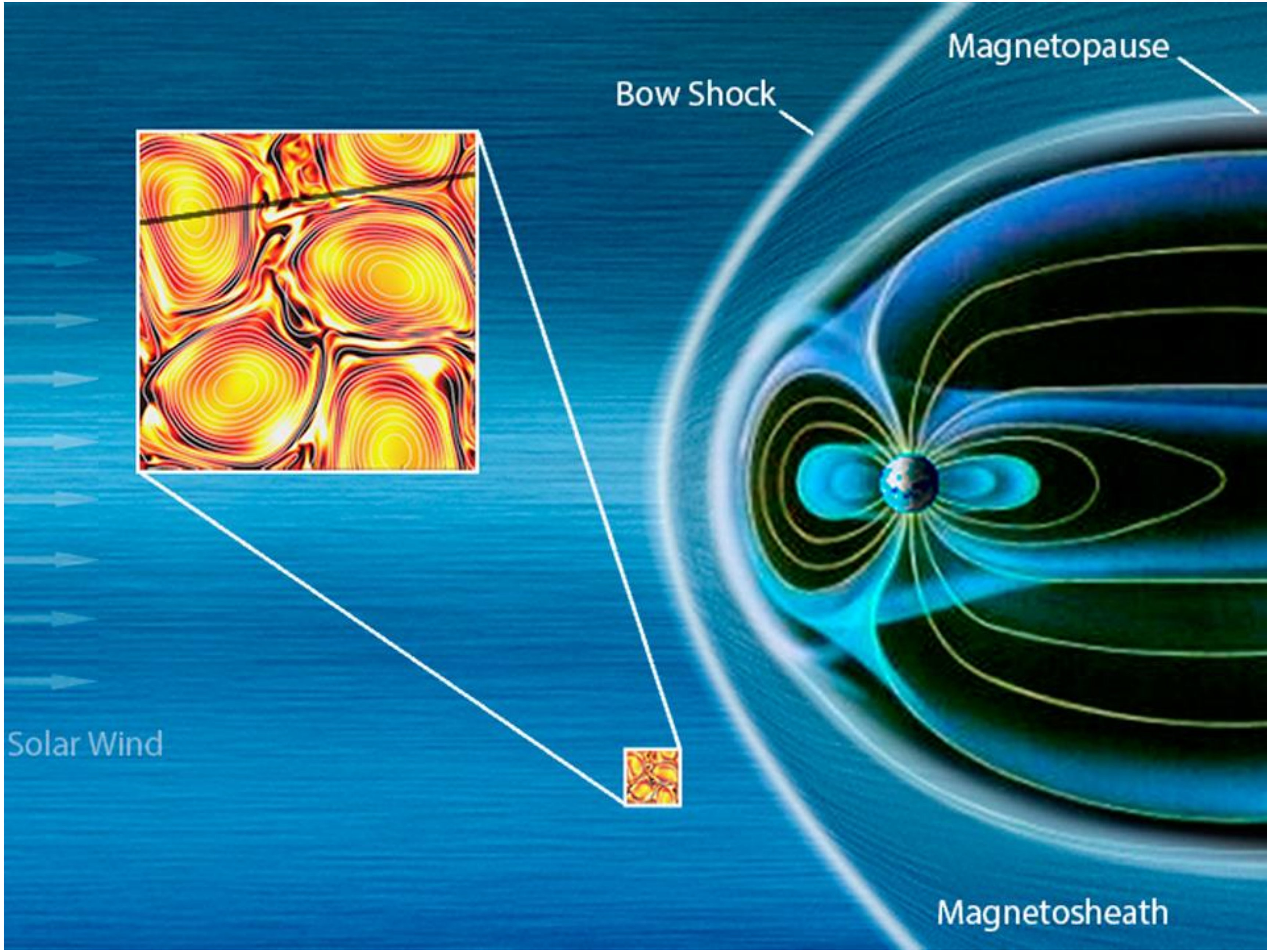
The magnetic field observed by Voyager 1 in the Local Interstellar Medium

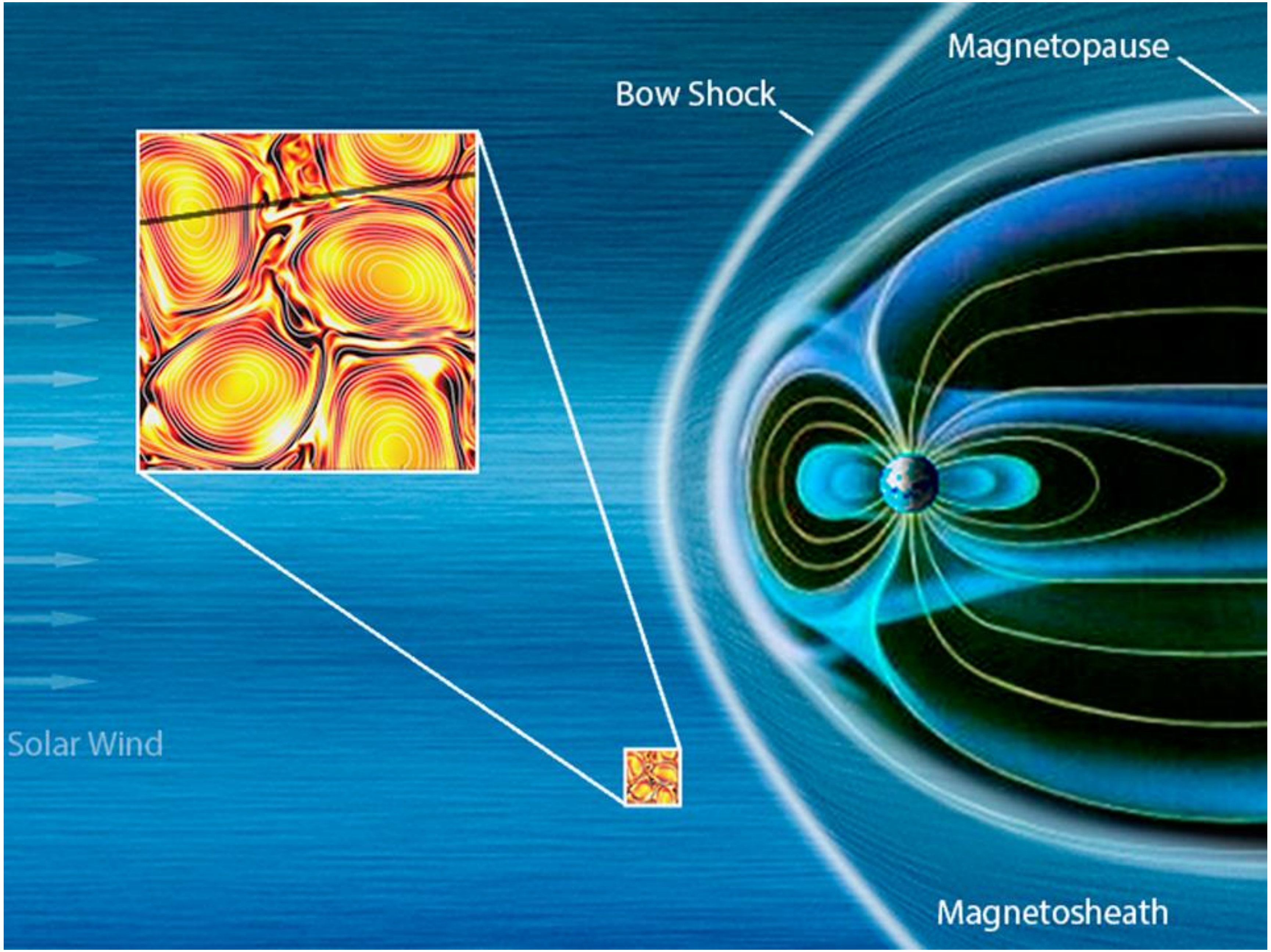




Burlaga et al., ApJ, 2015

- **Voyager 1 is measuring the short-wavelength end of the interstellar magnetic field turbulence spectrum**
- **Fluctuations are mostly in the *magnitude of the magnetic field***


Magnetopause
Bow Shock
Solar Wind
Magnetosheath


Magnetopause
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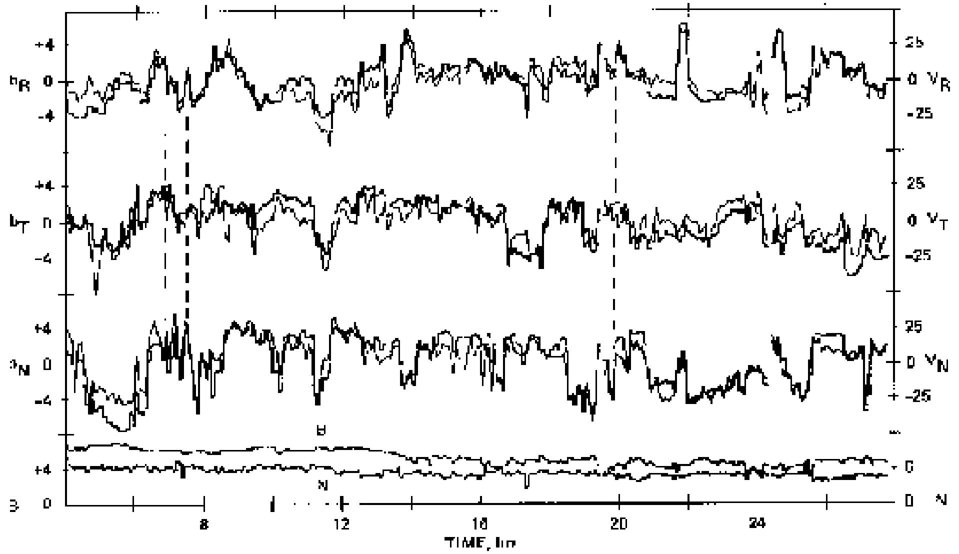
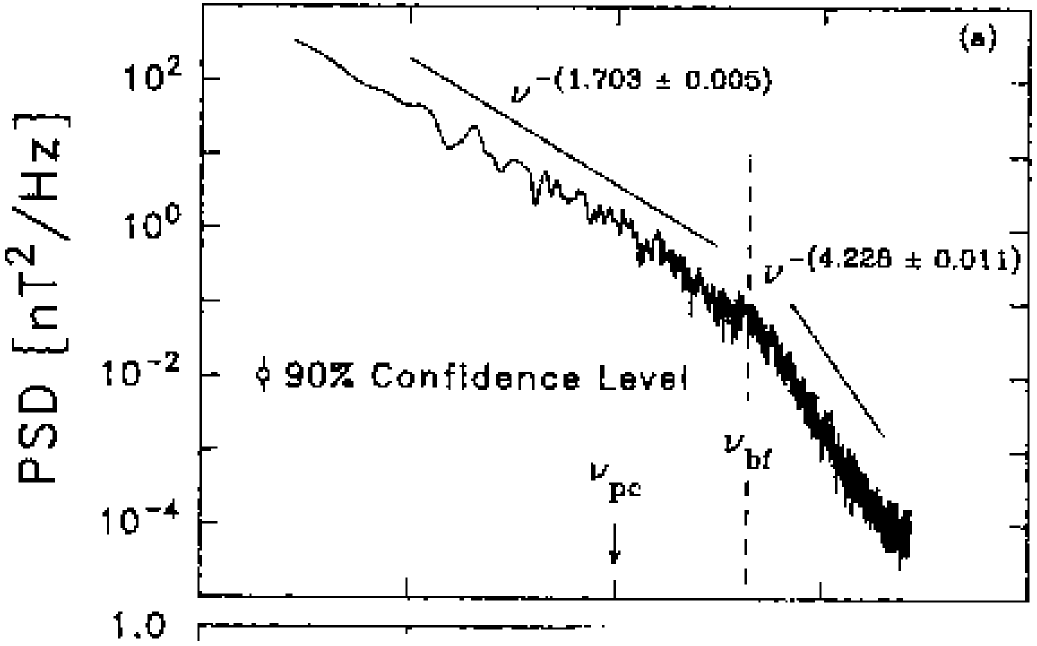
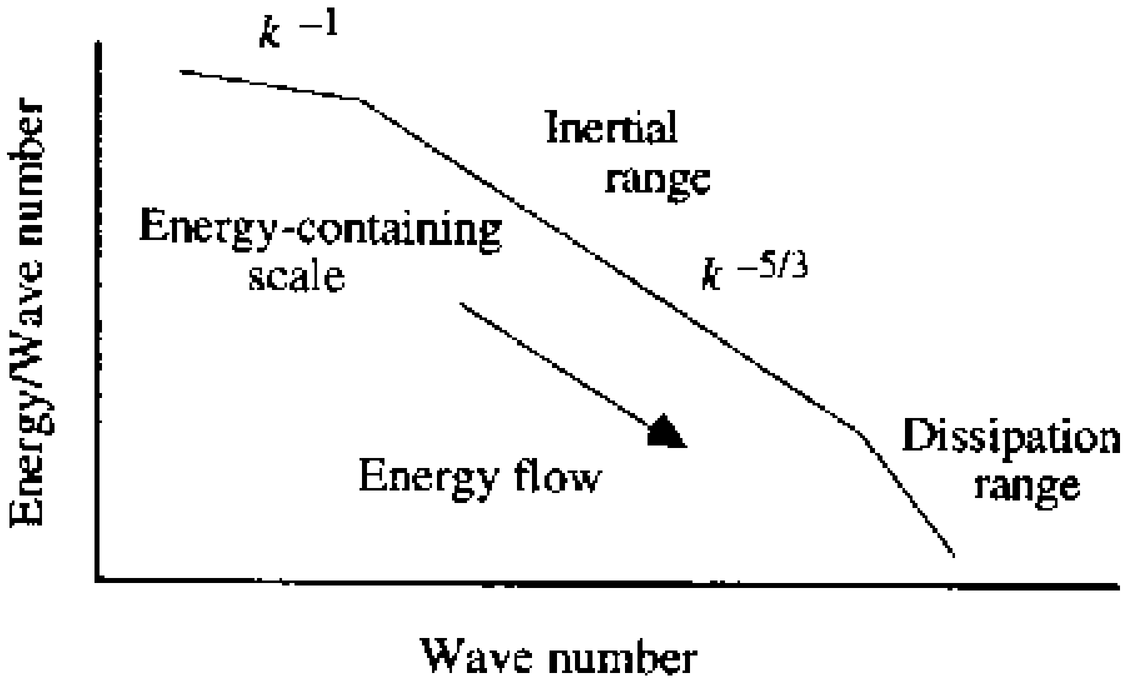
Observations of turbulence in the solar wind





Basic schematic of energy input, cascade to smaller scales, and dissipation



Turbulence

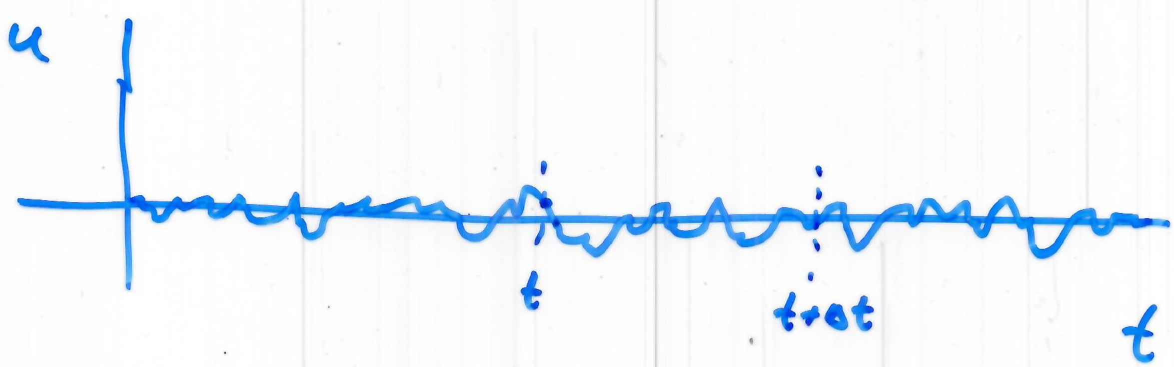
$$u(\underset{\sim}{r}, t) = \sum_{n=1}^{N_m} \delta \underset{\sim}{u}_n \, e^{i \underset{\sim}{k}_n \cdot \underset{\sim}{x} + i\phi_n}$$

if ϕ_n is randomly distributed $0 < \phi < 2\pi$

$\rightarrow$ get turbulence (if N_m is large)
(from waves)

this is the only concrete connection between waves & turbulence.

Consider a time series of observable data (velocity)



$\langle u \rangle = 0$ $\qquad$ $\langle \, \rangle \rightarrow$ "ensemble" average

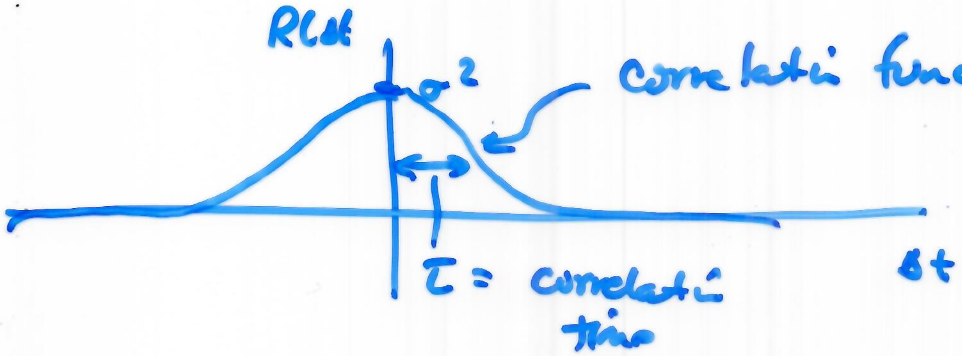
$\langle u^2 \rangle = \sigma^2 \leftarrow$ variance

Consider this function

$$\langle u(t)\, u(t+\Delta t)\rangle = R(\Delta t)$$

@ $\Delta t = 0$ $\qquad R(\Delta t) = \sigma^2$

@ $\Delta t \to \infty$ $\qquad R(\Delta t \to \infty) = 0$



a reasonable estimate of τ is

$$\tau = \frac{\int_{-\infty}^{\infty} R(\Delta t)\, \Delta t \; d\Delta t}{\int_{-\infty}^{\infty} R(\Delta t)\, d\Delta t}$$

correlation time

coherence time

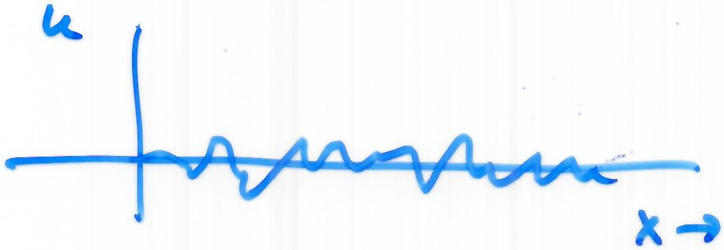
What about other scales?

Wiener-Khinchin theorem – the spectral decomposition of the correlation function is the power spectrum

$$P(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\Delta t)\, e^{-i\omega\Delta t}\, d\Delta t$$

fourier transform of corr. function

Can also do this in space



$$R(\Delta x) = \langle u(x)\, u(x+\Delta x)\rangle$$

$$P(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\Delta x)\, e^{-ik\Delta x}\, d\Delta x$$

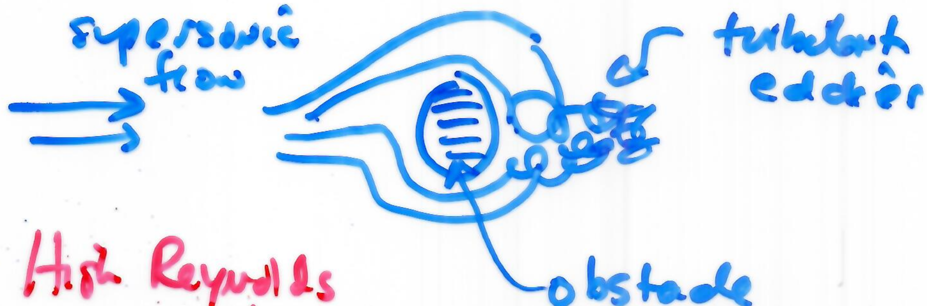
$$[P(k)] = u^2 L$$

$u^2 \propto$ energy

How does turbulence form?

plasma instabilities, supernovae explosions, stream-stream interactions, stirring coffee

"eddy"



High Reynolds #

"laminar" flow → usually no turbulence

usually subsonic

Low Reynolds #

Recall Reynolds # $R = \dfrac{|\rho \underset{\sim}{u} \cdot \nabla \underset{\sim}{u}|}{|\mu \nabla^2 \underset{\sim}{u}|}$

if u is large, typically R is large

$$= \frac{\rho u^2/L}{\mu \, u/L^2}$$

laminar flow, u is small R is small

$$\boxed{R = \frac{\rho L u}{\mu}}$$

if $R \gg 1$ → ~~the~~ viscosity has no effect.

⇒ cannot smooth out gradients (viscosity is a diffusion effect)

∴ turbulence best starts a large Reynolds #'s

→ high speed flows $R \propto u$

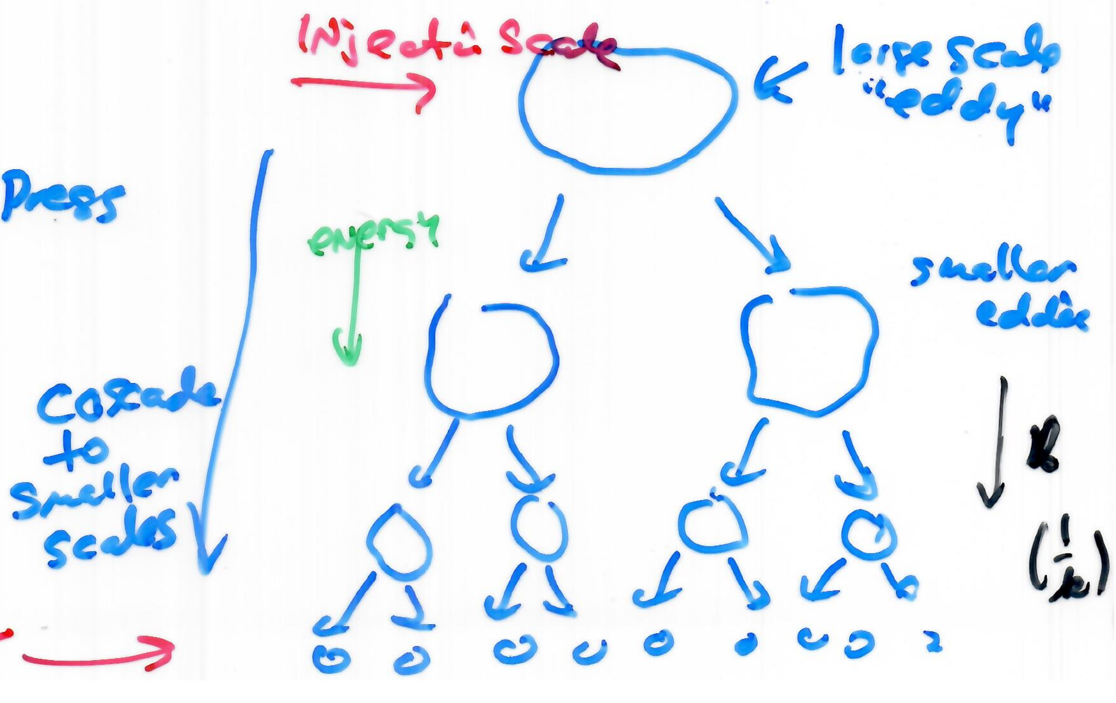
→ large scales $R \propto L$

at small scales, R is small → viscosity smooths out "dissipates" turbulence

turbulence → scale dependent.

Taylor, 1922
Cambridge U. Press
applied to weather

energy $\sim u^2$



How does this cascade happen in terms of wavelength? what is the distribution w/ scale?

Kolmogorov

postulated that $P(k)$ only depends on the energy cascade rate and the scale.

$$P(k) \propto (\dot{\epsilon})^{\alpha} k^{\beta}$$

$\dot{\epsilon} \rightarrow$ energy cascade rate

$k \rightarrow$ wave number

$\alpha, \beta \rightarrow$ t.b.d.

$$P(k) = C \, \dot{\epsilon}^{\alpha} k^{\beta}$$

$$[\dot{\epsilon}] = \frac{\text{energy}}{\text{time}} = \frac{V^2}{T} = \frac{L^2}{T^3}$$

$$[k] = \frac{1}{L}$$

$$[P(k)] = \text{energy} \cdot L = v^2 L$$

$$= \frac{L^3}{T^2}$$

$$\therefore \quad L^3 = L^{2\alpha} L^{-\beta} \qquad 3 = 2\alpha - \beta$$

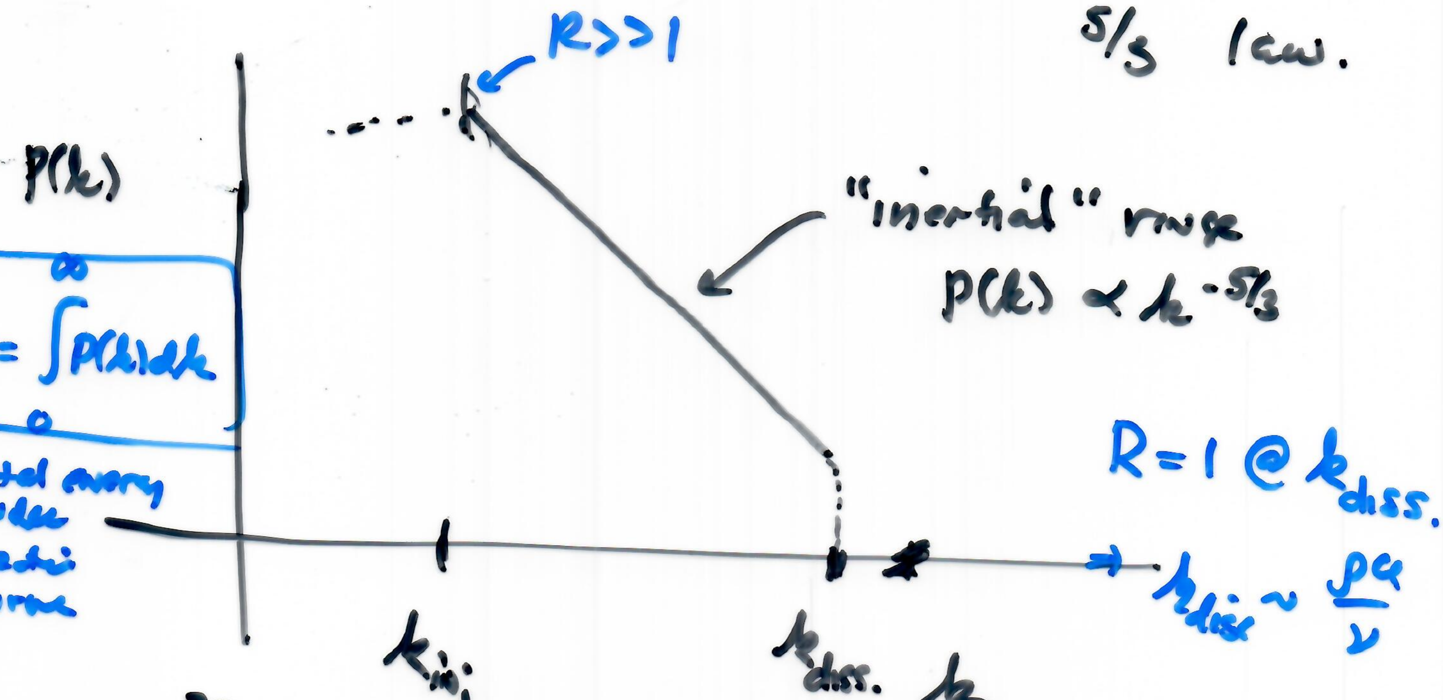
$$T^{-2} = T^{-3\alpha} \qquad -2 = -3\alpha \Rightarrow \alpha = 2/3$$

$$\Rightarrow \beta = 2(2/3) - 3$$

$$= 4/3 - 9/3 = -5/3$$

$$\therefore \quad \boxed{P(k) \propto (\dot{\epsilon})^{2/3} k^{-5/3}}$$

Kolmogorov "famous" 5/3 law.



$$\boxed{\sigma^2 = \int_0^\infty P(k)\,dk}$$

= total energy under spectral curve

$k_{inj} = \frac{2\pi}{\lambda_{inj}}$ $\quad \lambda_{inj} \rightarrow$ injection scale $\qquad k_{diss} = \frac{2\pi}{\lambda_{diss}}$