PTYS 558: Wednesday, April 29, 2020

- Finish derivation relating the spatial diffusion coefficient (along the magnetic field) to the pitch-angle diffusion coefficient.
- Discuss energy change
- Particle acceleration at shocks

The Parker Transport Equation:

$$\frac{\partial f}{\partial t} = -V_{w,i}\frac{\partial f}{\partial x_i} + \frac{\partial}{\partial x_i}\kappa_{ij}^{(S)}\frac{\partial f}{\partial x_j} - V_{D,i}\frac{\partial f}{\partial x_i} + \frac{1}{3}\frac{\partial V_{w,i}}{\partial x_i}\frac{\partial f}{\partial \ln p} + Q$$

advection ***diffusion*** ***drift*** ***energy change*** ***source***

Where the drift velocity due to the large scale curvature and gradient of the average magnetic field is:

$$\mathbf{V_d} = \frac{pcw}{3q}\,\nabla \times \left[\frac{\mathbf{B}}{B^2}\right]$$

The focus of today's lecture

And the symmetric part of the diffusion tensor is:

$$\kappa_{ij}^{(S)} = \kappa_\perp \delta_{ij} + (\kappa_\parallel - \kappa_\perp)\frac{B_i B_j}{B^2}$$

The Parker Transport Equation is valid whenever the anisotropy is small (as observed for GCRs). It is widely used and remarkably general.

A review of Monday's discussion

- We showed that magnetic fields with fluctuations on the scale of the gyroradius of the particles, cause the particles' pitch angles to vary in a manner that is consistent with "diffusion" – this is called pitch-angle diffusion.
- The pitch-angle diffusion coefficient is determined by the power spectrum of the turbulent magnetic field through quasi-linear theory
- We also showed that particles undergo a random walk in space, so the distribution can be described as a diffusion in space as well. This is the term in the Parker equation.
- Since the Parker equation is an average over pitch-angle – there is no pitch angle in this equation, we must relate the spatial diffusion to the pitch-angle diffusion separately in order to relate the spatial diffusion to the magnetic field power spectrum. This derivation started towards the end of the lecture on Monday

Particle Transport (contd)

Pitch angle diffusion & its relation to spatial diffusion

recall, we had.

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu}\left(D_{\mu\mu}\frac{\partial f}{\partial \mu}\right)$$

assume

$$f = \frac{1}{2} f_0 + f_1(\mu) \qquad f_0 \to \text{isotropic part}$$

$$\langle f \rangle = \int_{-1}^{1} f\, d\mu = f_0$$

$$\int_{-1}^{1} f_1\, d\mu = 0$$

substitute in the p.a. diff. eq.

$$\frac{1}{2}\frac{\partial f_0}{\partial t} + \frac{\partial f_1}{\partial t} + \frac{1}{2} v\mu \frac{\partial f_0}{\partial z} + v\mu\frac{\partial f_1}{\partial z} = \frac{\partial}{\partial \mu}\left(D_{\mu\mu}\frac{\partial f_1}{\partial \mu}\right) \qquad (1)$$

Integrate over μ from -1 to $+1$

$$\frac{\partial f_0}{\partial t} + v\int_{-1}^{1} \mu \frac{\partial f_1}{\partial z}\, d\mu = D_{\mu\mu}\frac{\partial f_1}{\partial \mu}\Big]_{-1}^{+1} = 0$$

$$\boxed{\frac{\partial f_0}{\partial t} = -v\int_{-1}^{1} \mu \frac{\partial f_1}{\partial z}\, d\mu} \qquad (2)$$

multiply (2) x 1/2, subtract from (1)

$$\overset{1}{\frac{\partial f_1}{\partial t}} + \overset{2}{\frac{1}{2} v\mu \frac{\partial f_0}{\partial z}} + \overset{3}{v\mu \frac{\partial f_1}{\partial z}} = \overset{4}{\frac{\partial}{\partial \mu}\left(D_{\mu\mu} \frac{\partial f_1}{\partial \mu}\right)} + \overset{5}{v \int_{-1}^{1} \mu \frac{\partial f_1}{\partial z} d\mu}$$

because T, characteristic time scale & L characteristic length scale are <u>large</u> compared to scattering time, τ, and mean-free path λ

$$T \gg \tau$$
$$L \gg \lambda$$

and $f_1 \ll f_0$

only terms 2 & 4 are important in this eq.

$$\therefore \quad \frac{1}{2} v\mu \frac{\partial f_0}{\partial z} \simeq \frac{\partial}{\partial \mu}\left(D_{\mu\mu} \frac{\partial f_1}{\partial \mu}\right)$$

$$\Rightarrow \int_{-1}^{\mu} \frac{1}{2} v\mu' \frac{\partial f_0}{\partial z} d\mu' = D_{\mu\mu} \frac{\partial f_1}{\partial \mu}$$

$$\Rightarrow \frac{v}{4}(\mu^2 - 1) \frac{\partial f_0}{\partial z} = D_{\mu\mu} \frac{\partial f_1}{\partial \mu}$$

$$\Rightarrow \quad \frac{\partial f_1}{\partial \mu} = \frac{v}{4}(\mu^2-1)\frac{1}{D_{\mu\mu}}\frac{\partial f_0}{\partial z}$$

$$f_1 = \int_{-1}^{\mu} \frac{v}{4}(\mu'^2-1)\frac{1}{D_{\mu\mu}(\mu')}\frac{\partial f_0}{\partial z}\,\mu'$$

Insert this into (2) (boxed eq.)

$$\frac{\partial f_0}{\partial z} = \frac{v^2}{4}\int_{-1}^{1}\mu\,d\mu\,\frac{\partial}{\partial z}\left[\int_{-1}^{\mu}\frac{1-\mu'^2}{D_{\mu\mu}}\frac{\partial f_0}{\partial z}\,d\mu'\right]$$

$$= \frac{\partial}{\partial z}\left(\frac{v^2}{4}\int_{-1}^{1}\mu\,d\mu\int_{-1}^{\mu}\frac{1-\mu'^2}{D_{\mu\mu}}\,d\mu'\;\frac{\partial f_0}{\partial z}\right)$$

$$= \frac{\partial}{\partial z}\,K\,\frac{\partial f_0}{\partial z}$$

where

$$K = \frac{v^2}{4}\int_{-1}^{1}\mu\,d\mu\int_{-1}^{\mu}\frac{1-\mu'^2}{D_{\mu\mu}(\mu')}\,d\mu'$$

Change order of integration to give

$$K = \frac{v^2}{4}\int_{-1}^{1} \frac{1-\mu'^2}{D_{\mu\mu}(\mu')}\, d\mu' \int_{\mu}^{1} \mu\, d\mu$$

$\vdots$

$$\boxed{K = \frac{v^2}{4}\int_{0}^{1} \frac{(1-\mu^2)^2\, d\mu}{D_{\mu\mu}}}$$

Earl, 1974
Lohmann, 1976

recall

$$\boxed{D_{\mu\mu} = \frac{\pi}{4}\Omega_0 (1-\mu^2)\, \frac{k_r P(k_r)}{B_0^2}}$$

quasi-linear theory
Jokipii, 1966

$$\boxed{k_r = \frac{\Omega}{v\mu}}$$

resonant wave #

$P \rightarrow$ power spectrum

$K = \frac{1}{3}\lambda v \leftarrow$ particle speed

$\leftarrow$ mean free path

Particle Transport: Energy change

recall eq. of motion of a charged particle

$$\frac{d\vec{p}}{dt} = q\underset{\sim}{E} + \frac{q}{c}\underset{\sim}{w} \times \underset{\sim}{B}$$

$$\underset{\sim}{w} \cdot \frac{d\underset{\sim}{p}}{dt} = q\,\underset{\sim}{w}\cdot\underset{\sim}{E}$$

$$\frac{dT}{dt} = q\,\underset{\sim}{w}\cdot\underset{\sim}{E} \qquad T = \text{Kinetic energy}$$

$$\underset{\sim}{E} = -\frac{1}{c}\underset{\sim}{U} \times \underset{\sim}{B} \rightarrow \text{Ideal MHD}$$

$$\Rightarrow \quad \frac{dT}{dt} = -\frac{q}{c}\underset{\sim}{w}\cdot(\underset{\sim}{U}\times\underset{\sim}{B})$$

T is in the lab frame - inertial frame.
Consider instead the frame moving w/ fluid

$$T' = T - \underset{\sim}{p}\cdot\underset{\sim}{U} + \frac{1}{2}mU^2$$

$$\Rightarrow \quad \frac{dT'}{dt} = \frac{dT}{dt} - \frac{d}{dt}(\underset{\sim}{p}\cdot\underset{\sim}{U}) + \frac{d}{dt}\left(\frac{1}{2}mU^2\right)$$

$$\frac{dT'}{dt} = -\frac{q}{c}\,\mathbf{w}\cdot(\mathbf{U}\times\mathbf{B}) - \mathbf{p}\cdot\frac{d\mathbf{U}}{dt} - \mathbf{U}\cdot\underbrace{\frac{d\mathbf{p}}{dt}}_{q\mathbf{E}+\frac{q}{c}\mathbf{w}\times\mathbf{B}} + m\mathbf{U}\cdot\frac{d\mathbf{U}}{dt}$$

and $q\mathbf{E}\cdot\mathbf{U} = -\frac{q}{c}\mathbf{U}\times\mathbf{B}\cdot\mathbf{U} = 0$

$$= -\frac{q}{c}\,\mathbf{w}\cdot(\mathbf{U}\times\mathbf{B}) - \mathbf{p}\cdot\frac{d\mathbf{U}}{dt} - \frac{q}{c}\,\mathbf{U}\cdot(\mathbf{w}\times\mathbf{B}) + m\mathbf{U}\cdot\frac{d\mathbf{U}}{dt}$$

these cancel after implementing a vector identity

$$= -(\mathbf{p} - m\mathbf{U})\cdot\frac{d\mathbf{U}}{dt}$$

$$\frac{dT'}{dt} = -\mathbf{p}'\cdot\frac{d\mathbf{U}}{dt}$$

$\frac{d}{dt} \rightarrow$ along particle trajectory

$$= -\mathbf{p}'\cdot\left(\frac{\partial\mathbf{U}}{\partial t} + \mathbf{w}\cdot\nabla\mathbf{U}\right) \longleftarrow$$ use convective derivative

Consider a collection of particles, distributed <u>isotropically</u>, then

$$\frac{d\langle T'\rangle}{dt} = \left\langle -\underset{\sim}{p}' \cdot \left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{w}' \cdot \nabla \underset{\sim}{u} \right) \right\rangle$$

avg. over isotropic dist.

$$= -\left\langle \underset{\sim}{p}' \cdot (\underset{\sim}{w}' \cdot \nabla \underset{\sim}{u}) \right\rangle$$

assume $\underset{\sim}{u}$ varies slowly in time compared to scattering time

index not.

$$= -\langle p_i' w_j' \partial_j u_i \rangle$$

$$= -\underbrace{\langle p_i' w_j' \rangle}_{\frac{1}{3} p' w' \delta_{ij}} \partial_j u_i$$

assume u varies in space slowly compared to λ (mean free path)

kronecker delta

$$= -\frac{1}{3} p' w' \nabla \cdot \underset{\sim}{u}$$

recall $dT'/dt = w' \, dp'/dt$

$$\Rightarrow \boxed{\frac{d\langle p\rangle}{dt} = -\frac{1}{3} p' \nabla \cdot \underset{\sim}{u}}$$

energy change of an isotropic dist. of particles

Cosmic-Ray Transport Equation

$$\frac{\partial f}{\partial t} + \underset{\sim}{U} \cdot \nabla f = \nabla \cdot (\underset{\approx}{K} \cdot \nabla f) + \frac{1}{3} \nabla \cdot \underset{\sim}{U} \frac{\partial f}{\partial \ln p} + S - L$$

advection w/ fluid

spatial diffusion <u>and</u> drifts

energy change

sources losses

where,

$$K_{ij} = K_\perp \delta_{ij} + (K_\parallel - K_\perp) \frac{B_i B_j}{B^2} + \epsilon_{ijk} K_A \frac{B_k}{B}$$

$K_\perp$ = cross-field diffusion coeff.

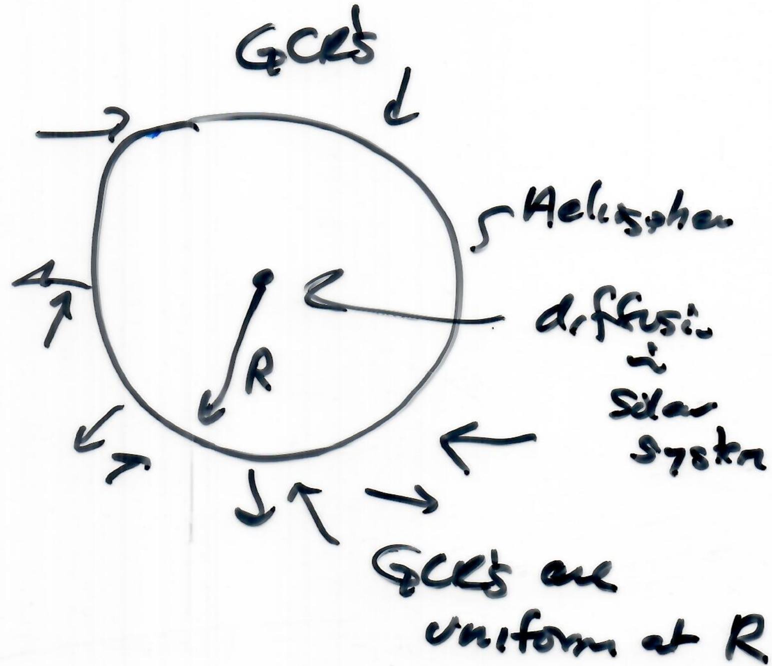
$K_\parallel$ = parallel " " (previous discussion)

K_A = anti-symmetric component

→ this is where drifts are!

GCR modulation

Consider the spherical coord. rep. of the CR transport eq. (Parker equation) in steady state



$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\kappa r^2 \frac{\partial f}{\partial r}\right) - U\frac{\partial f}{\partial r} - \frac{2Up}{3r}\frac{\partial f}{\partial p} = 0$$

this can be written

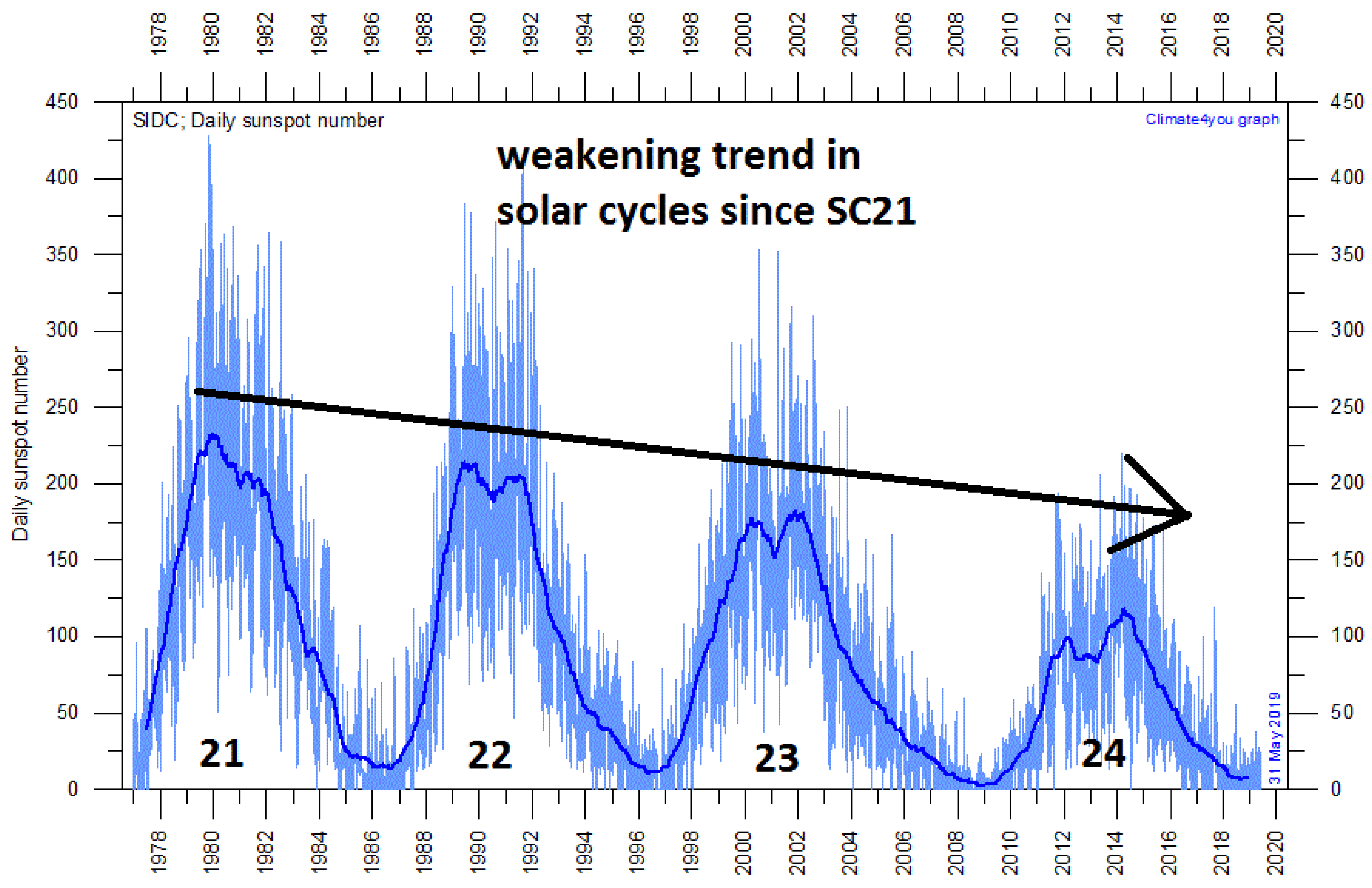
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\kappa \frac{\partial f}{\partial r} - Uf\right)\right] + \frac{2U}{3rp^2}\frac{\partial}{\partial p}\left(p^3 f\right) = 0$$

if one ignores the last term, we find

$$\boxed{f(r,p) = f(R,p)\, e^{-\int_r^R \frac{U\,dr'}{\kappa(r',p)}}}$$

commonly used "model" (not a good one, too)

Cosmic Rays on Earth Since 1964


SIDC; Daily sunspot number
Climate4you graph
weakening trend in
solar cycles since SC21
Daily sunspot number
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31 May 2019
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