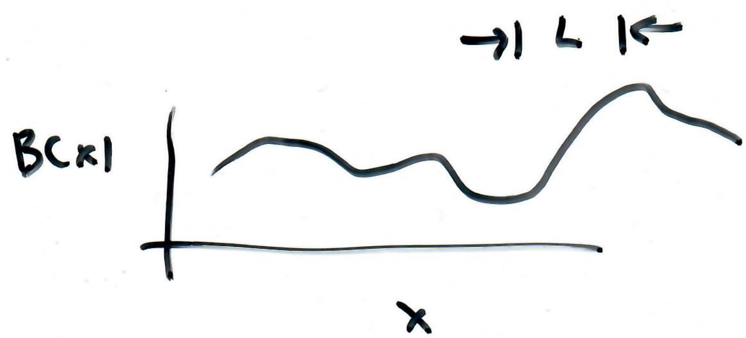
Orbit theory (cont.)

Consider a slowly varying mag. field of the form

$$\underset{\sim}{B}(x) = (0, 0, B(x))$$

$$\underset{\sim}{E} = 0$$



Consider "slowly" varying to mean $r_g << L$
where r_g = particle gyroradius

Equation of motion give

$$m\dot{v}_x = \frac{q}{c} v_y B(x)$$

$$m\dot{v}_y = -\frac{q}{c} v_x B(x)$$

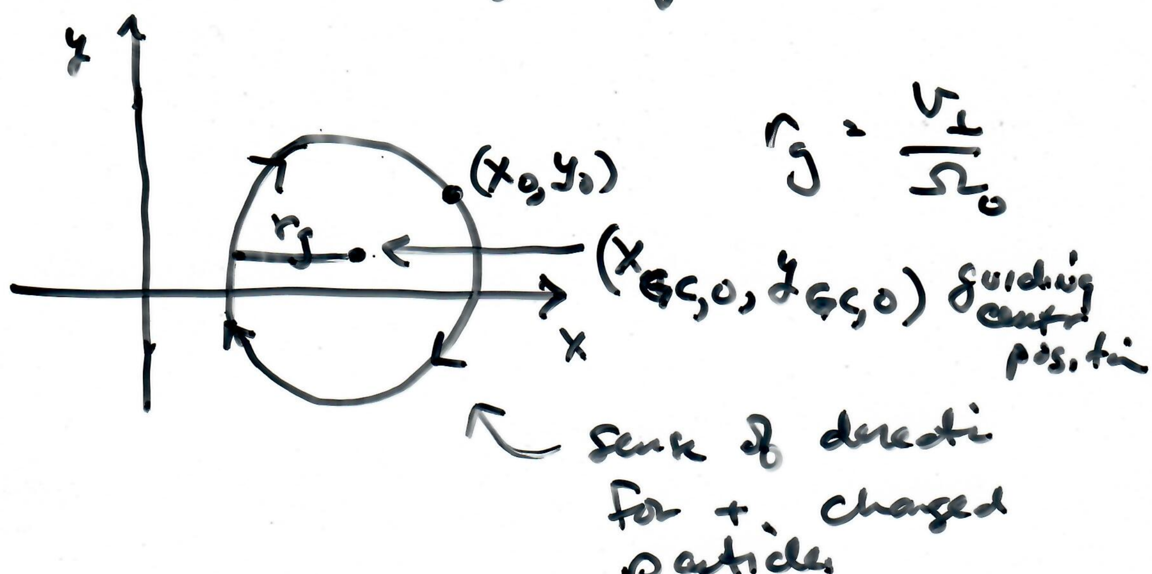
$$m\dot{v}_z = 0 \quad \Rightarrow \quad v_z = \text{constant}$$

rewrite as

$$\dot{v}_x = v_y \Omega(x) \qquad \text{where } \Omega(x) = \frac{q B(x)}{mc}$$

$$\dot{v}_y = -v_x \Omega(x)$$

Consider the particle "guiding center"



$$r_g = \frac{v_\perp}{\Omega_0}$$

one has, for the constant B case

$$x = x_0 + \frac{v_\perp}{\Omega}\left(\sin(\Omega_0 t - \phi_0) + \sin\phi_0\right)$$

note @ $t=0$ $x = x_0$

$$\phi_0 = \tan^{-1}\left(\frac{v_{y0}}{v_{x0}}\right)$$

Can re-write as

$$x = \underbrace{x_0 + \frac{v_\perp}{\Omega}\sin\phi_0}_{x_{G.C.,0}} + \frac{v_\perp}{\Omega}\sin(\Omega_0 t - \phi_0)$$

$$x = x_{G.C.,0} + \frac{v_\perp}{\Omega}\sin(\Omega_0 t - \phi_0)$$

returning to $\Omega(x)$. Varies slowly. Consider an expansion about $x = x_{G.C.\,0}$ (drop subscript! don't forget we are consider G.C.)

$$\Omega(x) = \Omega(x = x_0) + (x - x_0)\left.\frac{d\Omega}{dx}\right|_{x = x_0} + \cdots$$

$$\approx \Omega_0 + (x - x_0)\,\Omega'$$

Thus, we have

$\Omega' = d\Omega/dx$

$$\dot{v}_x = \Omega_0 v_y + (x - x_0)\,\Omega' v_y$$

small compared to first term

$$\dot{v}_y = -\Omega_0 v_x - (x - x_0)\,\Omega' v_x$$

we can use 0^{th} order solutions here.

$$(x - x_0)\,\Omega' \ll \Omega_0$$

$$\Omega' = \left.\frac{d\Omega}{dx}\right|_{x = x_0} \sim \frac{\Omega_0}{L}$$

$\Rightarrow \dfrac{x - x_0}{L} \ll 1$, the approx. we use

the 0^{th}-order solutions are:

$$\left.\begin{aligned} x - x_0 &= v_\perp/\Omega_0 \sin(\Omega_0 t - \phi_0) \\ v_y &= -v_\perp \sin(\Omega_0 t - \phi_0) \\ v_x &= v_\perp \cos(\Omega_0 t - \phi_0) \end{aligned}\right\}$$

solution for constant B case.

we find

$$\dot{v}_x = \Omega_0 v_y + \left[\frac{v_\perp}{\Omega_0}\sin(\Omega_0 t - \phi_0)\right]\Omega'\left[-v_\perp \sin(\Omega_0 t - \phi_0)\right]$$

$$= \Omega_0 v_y - \frac{v_\perp^2 \Omega'}{\Omega_0}\sin^2(\Omega_0 t - \phi_0)$$

$$\dot{v}_y = -\Omega_0 v_x - \left[\frac{v_\perp}{\Omega_0}\sin(\Omega_0 t - \phi_0)\right]\Omega'\left[v_\perp \cos(\Omega_0 t - \phi_0)\right]$$

$$= -\Omega_0 v_x - \frac{v_\perp^2 \Omega'}{\Omega_0}\sin(\Omega_0 t - \phi_0)\cos(\Omega_0 t - \phi_0)$$

it is convenient to define a new variable

$$\xi = x + iy$$

$$\dot{\xi} = \dot{x} + i\dot{y} = v_x + i v_y$$

$$(\text{note: } -i\dot{\xi} = v_y - i v_x)$$

$$\ddot{\xi} = \ddot{x} + i\ddot{y} = \dot{v}_x + i\dot{v}_y$$

In terms of ξ, we have

$$\ddot{\xi} = \Omega_0 v_y - \frac{v_\perp^2 \Omega'}{\Omega_0}\sin^2(\Omega_0 t - \phi_0)$$
$$+ i\left[-\Omega_0 v_x - \frac{v_\perp^2 \Omega'}{\Omega_0}\sin(\Omega_0 t - \phi_0)\cos(\Omega_0 t - \phi_0)\right]$$

$$\Rightarrow \quad \ddot{\xi} = -i\dot{\xi}\,\Omega_0 - \frac{v_\perp^2\Omega'}{\Omega_0}\sin(\Omega_0 t - \phi_0)\left[\sin(\Omega_0 t - \phi_0) + i\cos(\Omega_0 t - \phi_0)\right]$$

$$= -i\dot{\xi}\,\Omega_0 - \frac{v_\perp^2\Omega'}{\Omega_0}\sin(\Omega_0 t - \phi_0)\, i\, e^{-i(\Omega_0 t - \phi_0)}$$

$$\Rightarrow \quad \ddot{\xi} + i\Omega_0\dot{\xi} = -i\frac{v_\perp^2\Omega'}{\Omega_0}\sin(\Omega_0 t - \phi_0)\, e^{-i(\Omega_0 t - \phi_0)}$$

Consider the left-hand side. Note that

$$e^{-i\Omega_0 t}\frac{d}{dt}\left(e^{i\Omega_0 t}\dot{\xi}\right) = \ddot{\xi} + i\Omega_0\dot{\xi}$$

$$\Rightarrow \quad e^{-i\Omega_0 t'}\frac{d}{dt}\left(e^{i\Omega_0 t}\dot{\xi}\right) = -i\frac{v_\perp^2\Omega'}{\Omega_0}\sin(\Omega_0 t - \phi_0)\, e^{-i\Omega_0 t'}\, e^{+i\phi_0}$$

integrate over time, to give

$$e^{i\Omega_0 t}\dot{\xi} - \dot{\xi}(0) = -i\frac{v_\perp^2\Omega'}{\Omega_0}e^{i\phi_0}\int_0^t dt'\sin(\Omega_0 t' - \phi_0)$$

$$\Rightarrow \quad \dot{\xi} = \dot{\xi}(0)\,e^{-i\Omega_0 t} - i\frac{v_\perp^2\Omega'}{\Omega_0}e^{-i(\Omega_0 t - \phi_0)}\int_0^t dt'\sin(\;)$$

$$= \dot{\xi}(0)\,e^{-i\Omega_0 t} - i\frac{v_\perp^2\Omega'}{\Omega_0}e^{-i(\Omega_0 t - \phi_0)}\left(-\frac{1}{\Omega_0}\cos(\Omega_0 t' - \phi_0)\right)\Big|_0^t$$

$$\dot{\jmath} = \dot{\jmath}(0)\, e^{-i\Omega_0 t} + i \frac{v_\perp^2 \Omega'}{\Omega_0^2} e^{-i(\Omega_0 t - \phi_0)}$$

$$\times \left\{ \cos(\Omega_0 t - \phi_0) - \cos\phi_0 \right\}$$

can show $\quad \dot{\jmath}(0) = v_\perp e^{i\phi_0}$

where $\phi_0 = \tan^{-1}\left(\frac{v_{y0}}{v_{x0}}\right)$

$$\dot{\jmath} = v_\perp e^{-i(\Omega_0 t - \phi_0)} + i \frac{v_\perp^2 \Omega'}{\Omega_0^2} e^{-i(\Omega_0 t - \phi_0)} \left(\cos(\Omega_0 t - \phi_0) - \cos\phi_0\right)$$

look @ $v_y \rightarrow$ imaginary part of $\dot{\jmath}$

$$v_y = \mathrm{Im}(\dot{\jmath})$$

$$v_y = -v_\perp \sin(\Omega_0 t - \phi_0) + \frac{v_\perp^2 \Omega'}{\Omega_0^2}\Big[\cos^2(\Omega_0 t - \phi_0) - \cos(\Omega_0 t - \phi_0)\cos\phi_0\Big]$$

$$= -v_\perp \sin(\Omega_0 t - \phi_0) + \frac{v_\perp^2 \Omega'}{\Omega_0^2}\left\{ \frac{1}{2} + \frac{1}{2}\cos(2(\Omega_0 t - \phi_0)) - \cos(\Omega_0 t - \phi_0)\cos\phi_0 \right\}$$

this can be re-written slightly as

$$v_y = -v_\perp \sin(\Omega t - \phi_0) + v_\perp \underbrace{\left(\frac{r_g \Omega'}{\Omega_0}\right)}_{\mathcal{O}(r_g/L)} \left[-\frac{1}{2} + \frac{1}{2}\cos 2(\Omega_0 t - \phi_0) - \cos(\Omega_0 t - \phi_0)\cos\phi_0 \right]$$

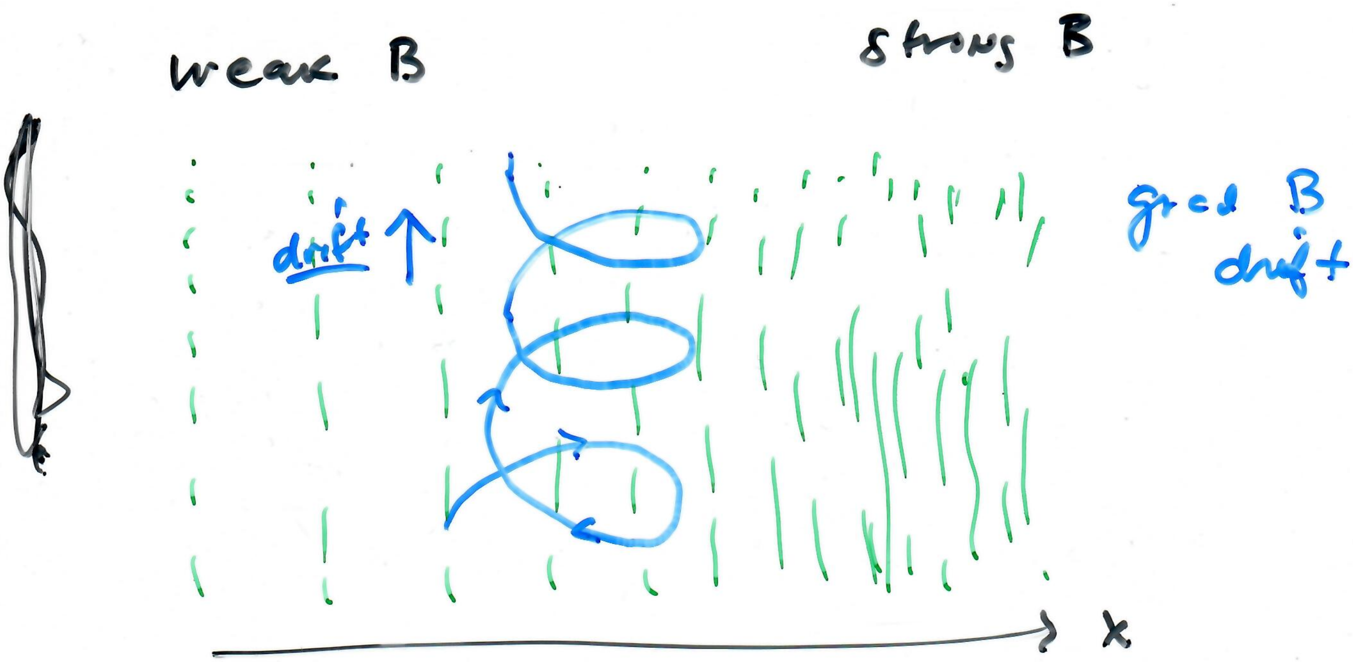
$$r_g = v_\perp / \Omega_0$$

note: there is a non-oscillatory component!

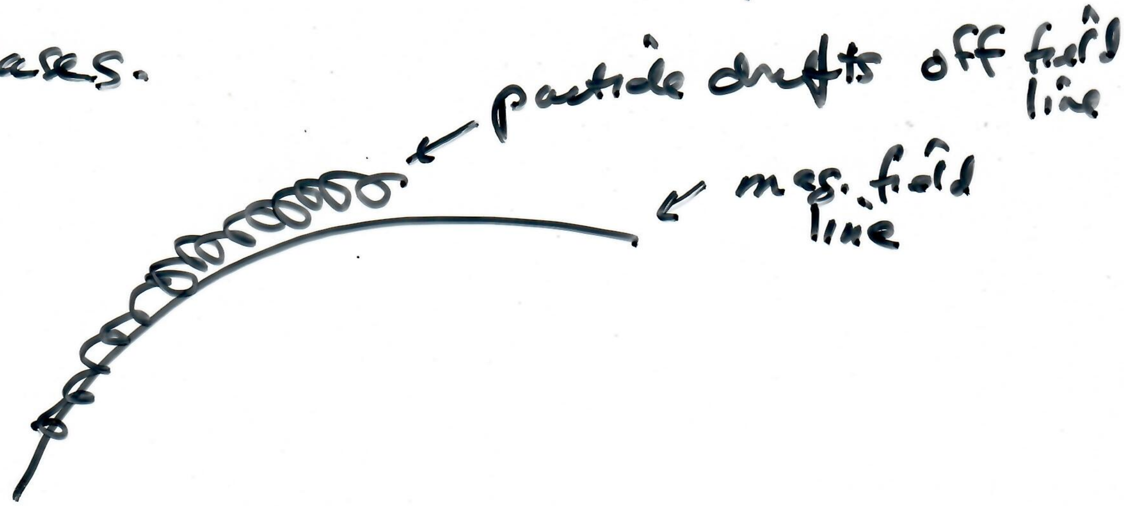
this represents a <u>drift speed</u>.

what causes this?

magnitude variation



there is also "curvature" drifts in some cases.



the general form of the "guiding-center" drift is

$$\boxed{\begin{aligned} \underset{\sim}{v}_{G.C.} &= \left[v_{\parallel} + \frac{c W_{\perp}}{2qB} \hat{b} \cdot (\nabla \times \hat{b}) \right] \hat{b} \\ &\quad + \frac{c W_{\perp}}{2qB} \hat{b} \times \nabla |\underset{\sim}{B}| + \frac{c W_{\parallel}}{qB} \hat{b} \times (\hat{b} \cdot \nabla) \hat{b} \end{aligned}}$$

$$\hat{b} = \underset{\sim}{B} / |\underset{\sim}{B}|$$

$$W_{\perp,\parallel} = \tfrac{1}{2} m v_{\perp,\parallel}^2$$