Orbit theory (cont.)

for the special case in which $\nabla \times \underset{\sim}{B} = 0$

$$\underset{\sim}{V}_{\substack{\text{GradB}\\ \text{drift}}} = \frac{c W_\perp}{q B^3} \underset{\sim}{B} \times \nabla \left| \underset{\sim}{B} \right| \qquad \text{grad-B drift}$$

$$\underset{\sim}{V}_{\substack{\text{curv.}\\ \text{drift}}} = \frac{2c W_\parallel}{q B^3} \underset{\sim}{B} \times \nabla \left| \underset{\sim}{B} \right| \qquad \text{curvature drift}$$

for the problem we did

$$\underset{\sim}{B} = B(x)\, \hat{z}$$

$$\underset{\sim}{B} \times \nabla \left| \underset{\sim}{B} \right| = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & B \\ \frac{dB}{dx} & 0 & 0 \end{vmatrix} = B \frac{dB}{dx} \hat{y}$$

$$\therefore \underset{\sim}{V}_G = \frac{c W_\perp}{q B^3} B \frac{dB}{dx} \hat{y} \qquad B' = dB/dx$$

$$= \frac{c \frac{1}{2} m v_\perp^2}{q B^2} B' \hat{y} = \frac{mc}{qB} \frac{1}{2} v_\perp^2 \frac{B'}{B} \hat{y}$$

re-write as

$$\delta_t \mu_m = \frac{1}{B}\,\delta_t w_\perp - \frac{w_\perp}{B^2}\,\delta_t B \quad (*)$$

recall eq. of motion

$$m\dot{\mathbf{v}} = q\mathbf{E} + \frac{q}{c}\,\mathbf{v}\times\mathbf{B}$$

let $\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel$

$$m\dot{\mathbf{v}}_\perp = \cancel{q\mathbf{E}_\perp}\; q\mathbf{E} + \frac{q}{c}\,\mathbf{v}_\perp\times\mathbf{B}$$

$$= q\mathbf{E} + \frac{q}{c}\,\mathbf{v}_\perp\times\mathbf{B}$$

$$\therefore \quad m\,\mathbf{v}_\perp\cdot\dot{\mathbf{v}}_\perp = q\,\mathbf{v}_\perp\cdot\mathbf{E} + \frac{q}{c}\,\cancelto{0}{\mathbf{v}_\perp\cdot(\mathbf{v}_\perp\times\mathbf{B})}$$

$$\frac{d}{dt}(w_\perp) = q\,\mathbf{v}_\perp\cdot\mathbf{E}$$

$$w_\perp = \frac{1}{T}\int_0^T q\,\mathbf{v}_\perp\cdot\mathbf{E}\,dt \qquad T = \frac{2\pi}{\Omega}$$

$$\underset{\sim}{v}_\xi = \Omega^{-1} \frac{1}{2} v_\perp^2 \frac{B'}{B} \hat{y}$$

$$= \frac{1}{2} v_\perp^2 \frac{\Omega'}{\Omega^2} \hat{y}$$ which is what we had before

<u>New topic</u>

Adiabatic Invariants : the magnetic moment.

$$\mu_M = \frac{W_\perp}{B} \simeq \text{constant}$$

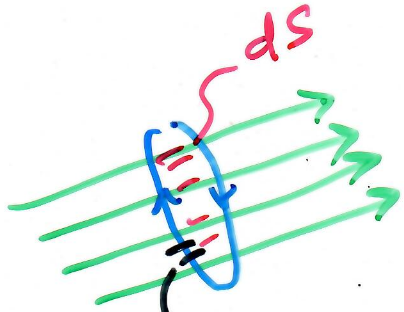
this is <u>very</u> good approximation when $r_g << L$

this is true over a "gyro orbit" not necessarily at ALL times (e.g. at a shock).

To show its invariance, consider

$$\frac{d}{dt}(\mu_m) = \frac{d}{dt}\left(\frac{W_\perp}{B}\right) = \frac{1}{B}\frac{dW_\perp}{dt} - \frac{W_\perp}{B^2}\frac{dB}{dt}$$

really, it is best to think of this as an average over a gyro orbit



$$d\underset{\sim}{l} = -\underset{\sim\perp}{v}\, dt$$

$$\therefore\ \delta_t W_\perp = \frac{-1}{T}\oint |q|\, \underset{\sim}{E}\cdot d\underset{\sim}{l}$$

$$= \frac{-|q|}{T}\oint \underset{\sim}{E}\cdot d\underset{\sim}{l}$$

Stokes theorem

$$= \frac{-|q|}{T}\int (\nabla\times\underset{\sim}{E})\cdot d\underset{\sim}{S}$$

$$= -\frac{|q|}{T}\int \left(-\frac{1}{c}\frac{\partial \underset{\sim}{B}}{\partial t}\right)\cdot d\underset{\sim}{S}$$

$$= \frac{|q|}{Tc}\int \frac{\partial \underset{\sim}{B}}{\partial t}\cdot d\underset{\sim}{S}$$

$$= \frac{|q|}{Tc}\,\pi r_g^2\, \delta_t B$$

$$\delta W_\perp = \frac{|q| \pi r_g^2}{Tc} \delta_t B$$ substitute into (*)

$$\therefore \quad \delta_t \mu_M = \frac{1}{B} \frac{|q| \pi r_g^2}{Tc} \delta_t B - \frac{W_\perp}{B^2} \delta_t B$$

$$T = 2\pi/\Omega = \frac{2\pi}{(|q|B/mc)} = \frac{2\pi mc}{|q|B}$$

plug into above, we find

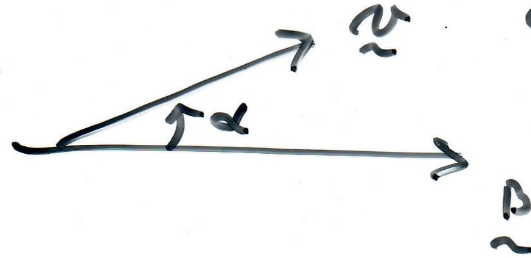
$$\delta_t \mu_M = 0 \rightarrow \mu_M \simeq \text{constant.}$$

$$\therefore \quad \frac{W_\perp}{B} = \text{constant}$$

<u>example</u>. charged particle moving in a varying B field, but not electric field

$$\therefore \quad v = \text{constant} \qquad \left(v = (v_\parallel^2 + v_\perp^2)^{1/2} \right)$$

$$\therefore \quad W_\perp = \tfrac{1}{2} m v_\perp^2 = \tfrac{1}{2} m v^2 \sin^2 \alpha$$



α = pitch angle

$v_\parallel = v \cos \alpha$

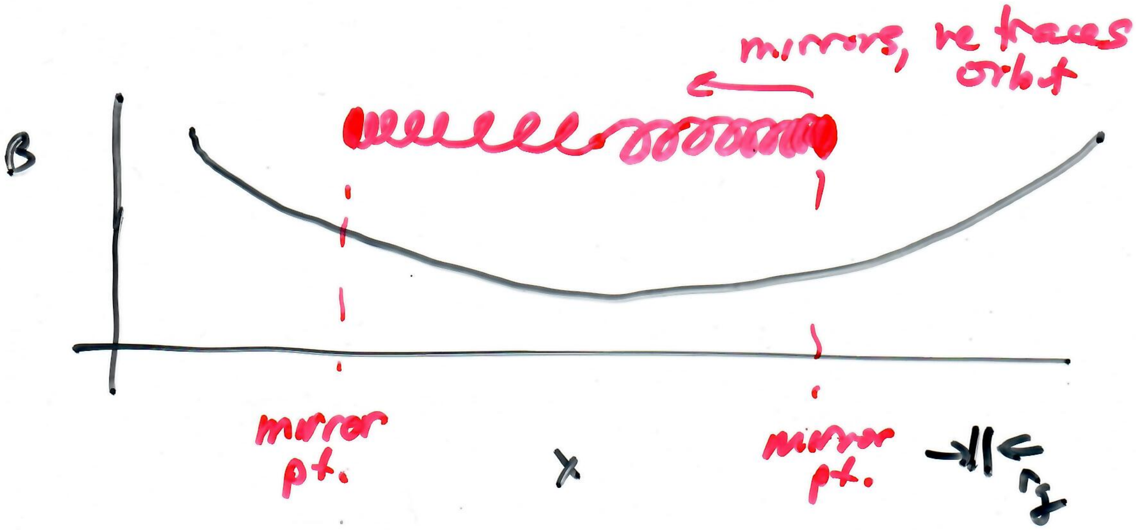
$v_\perp = v \sin \alpha$

$$\therefore \quad \frac{w_\perp}{B} = \text{const.} \Rightarrow \frac{\sin^2\alpha}{B} = \text{constant}$$

if $B = B_0$ when $\alpha = \alpha_0$, we have

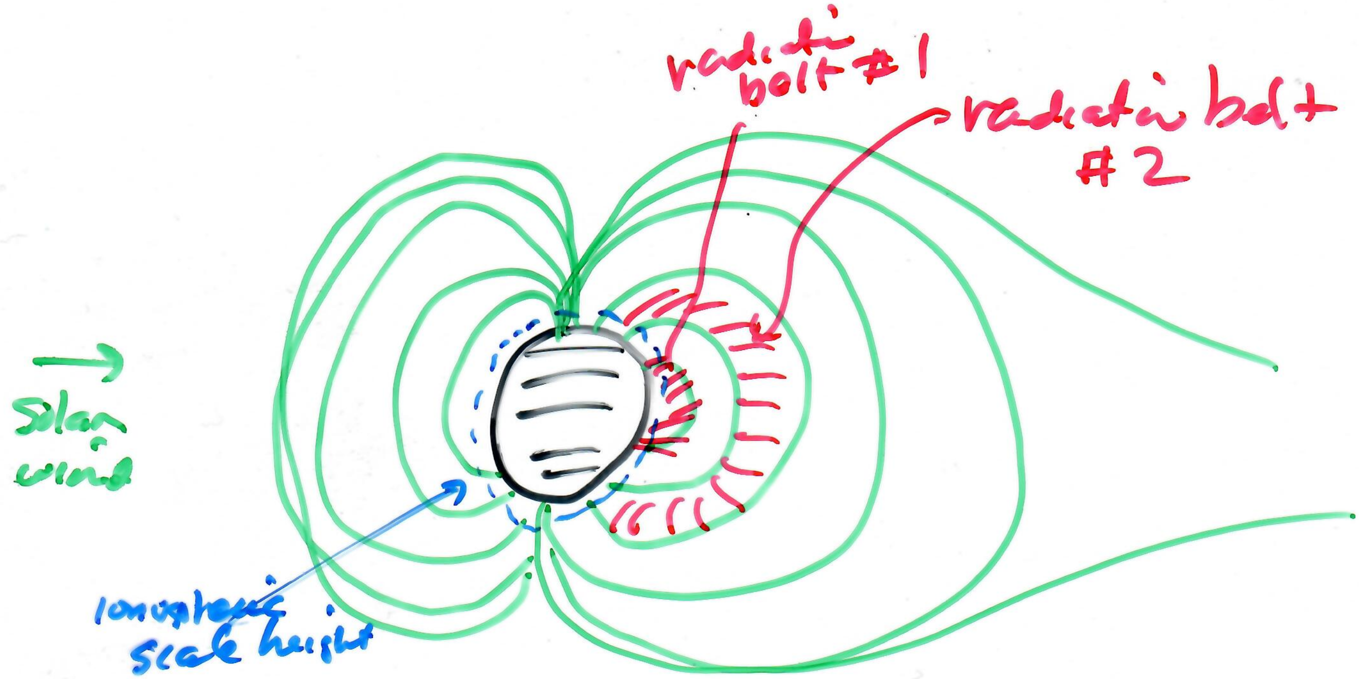
$$\frac{\sin^2\alpha}{B} = \frac{\sin^2\alpha_0}{B_0}$$

$$\Rightarrow \quad \sin\alpha = \sin\alpha_0 \left(\frac{B}{B_0}\right)^{1/2}$$



$r_g \ll L$

example magnetosphere



radiation belt #1 ⟹ source population is "CRAND"

CR → cosmic ray

A → albedo

ND → neutron decay

radiation belt #2 → mostly solar wind protons.

trapped particles also drift equatorially leading to a "ring current"

Another example: CME