Distribution function

$$f(\underset{\sim}{p}, \underset{\sim}{x}, t) = \frac{\text{particles}}{\underbrace{d^3\underset{\sim}{p}\ d^3\underset{\sim}{x}}}$$

"phase-space" volume element

total # of particle

$$N = \int f\, d^3\underset{\sim}{p}\, d^3\underset{\sim}{x}$$

number density

$$n = \int f\, d^3\underset{\sim}{p}$$

"normalization" of the dist. function

$$n = n(\underset{\sim}{x}, t)$$

in Cartesian coordinates

$$n(\underset{\sim}{x}, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p_x, p_y, p_z, \underset{\sim}{x}, t)\, dp_x\, dp_y\, dp_z$$

in spherical coord's

α = pitch angle
ϕ = phase angle
p = mag. momentum

$$n(\underset{\sim}{x}, t) = \int_0^{2\pi} d\phi \int_0^{\pi} \cos\alpha\, d\alpha \int_0^{\infty} p^2 dp\ f(p, \alpha, \phi, \underset{\sim}{x}, t)$$

Can also do this in terms of velocity, i.e.

$$n(\underset{\sim}{x}, t) = \int d^3\underset{\sim}{v}\; f(\underset{\sim}{v}, \underset{\sim}{x}, t)$$

But, be aware that the units of $f(\underset{\sim}{v}, \underset{\sim}{x}, t)$ are different from $f(\underset{\sim}{p}, \underset{\sim}{x}, t)$, because, it is also true

$$n(\underset{\sim}{x}, t) = \int \underbrace{d^3\underset{\sim}{p}}_{m^3 d^3\underset{\sim}{v}}\; f(\underset{\sim}{p}, \underset{\sim}{x}, t)$$

$$\therefore \; [f(\underset{\sim}{p}, \underset{\sim}{x}, t)] = \frac{1}{\text{mass}^3}[f(\underset{\sim}{v}, \underset{\sim}{x}, t)]$$

an example: $f(\overset{\text{energy}}{E}) \sim e^{-E/kT}$

k = Boltzmann constant
T = temperature

$f(E) \rightarrow$ Maxwell Boltzmann distr.

$$f(p) = C\, e^{-\frac{p^2}{2mkT}} \qquad \text{non. rel.}$$

$$n = 4\pi \int_0^\infty C\, e^{-\frac{p^2}{2mkT}}\, p^2\, dp$$

we find $\quad C = \dfrac{n}{(2\pi m kT)^{3/2}}$

$$f(p) = \frac{n}{(2\pi m kT)^{3/2}} e^{-\frac{p^2}{2mkT}}$$

an aside, in Cartesian coord., we have

$$n = \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z \; C \, e^{-\frac{p_x^2 + p_y^2 + p_z^2}{2mkT}}$$

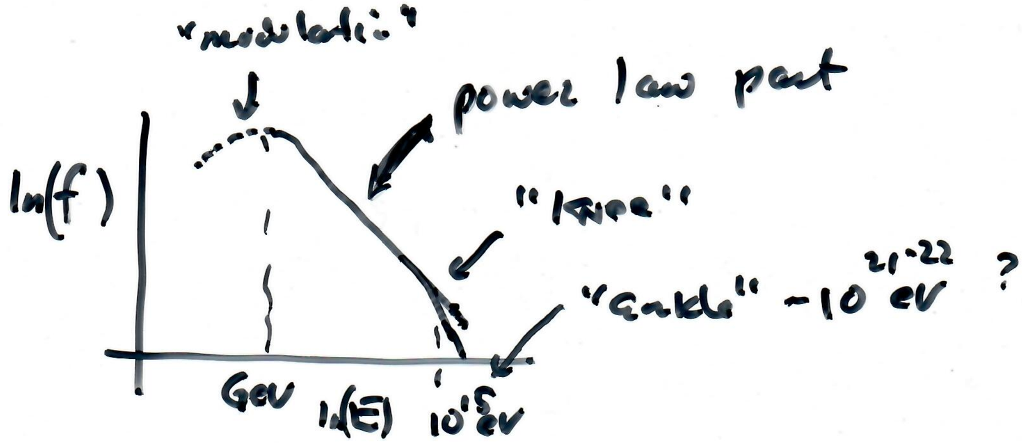
$$= C \left(\int_{-\infty}^{\infty} dp' \, e^{-\frac{p'^2}{2mkT}} \right)^3$$

we get $\quad C = \dfrac{n}{(2\pi m kT)^{3/2}}$

Another useful distribution is a "power law"

$f \sim E^{-\gamma} \qquad \gamma$ = spectral index

example cosmic ray spec.



in this case, we have

$$f(p) \propto p^{-\delta} = c\, p^{-\delta} \qquad \underline{\underline{p > p_0}}$$

$$n = 4\pi c' \int_{p_0}^{\infty} p^{-\delta}\, p^2\, dp$$

$$= -\frac{4\pi c'}{3-\delta}\, p_0^{3-\delta}$$

$$= \frac{4\pi c'}{\delta - 3}\, p_0^{3-\delta} \qquad \text{note } \delta > 3!$$

there is an even more stringent restriction on δ if one considers the energy.

other useful ~~stuff~~ bulk properties

$$\underset{\sim}{u}(\underset{\sim}{x}, t) = \frac{1}{n} \int d^3\underset{\sim}{p}\; f(\underset{\sim}{p}, \underset{\sim}{x}, t)\, \underset{\sim}{v} \qquad \underset{\sim}{p} = m\underset{\sim}{v}$$

$$\mathcal{E} = \text{energy density} = \int d^3\underset{\sim}{p}\; E\, f(\underset{\sim}{p}, \underset{\sim}{x}, t)$$

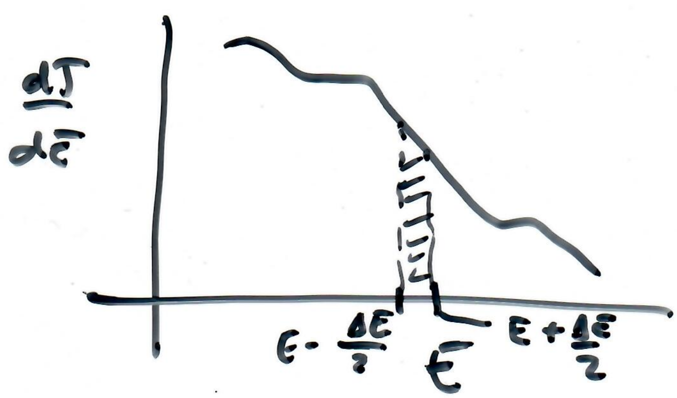
$$= \int d^3\underset{\sim}{p}\; \frac{p^2}{2m} f(\underset{\sim}{p}, \underset{\sim}{x}, t)$$

also, another form of the distribution function is the "differential intensity"

$$\frac{dJ}{dE} = p^2 f$$

common quantity plotted by observers.

it has units of flux / energy / solid angle



this is called an "Energy Spectrum" (space plasma physics)

$$\left[\frac{dJ}{dE}\right] = \frac{\text{part.}}{\text{cm}^2 \cdot \text{s} \cdot \text{sr} \cdot \text{MeV}}$$

typically

how does f vary in space & time?

consider

$$f(\mathbf{p}+\Delta\mathbf{p},\ \mathbf{x}+\Delta\mathbf{x},\ t+\Delta t) - f(\mathbf{p},\mathbf{x},t)$$

$$= (\Delta f)_{\text{collisions}}$$

$$\cancel{f(\mathbf{p})} + \Delta\mathbf{p}\cdot\frac{\partial f}{\partial \mathbf{p}}\bigg|_{\mathbf{p},\mathbf{x},t} + \Delta\mathbf{x}\cdot\frac{\partial f}{\partial \mathbf{x}}\bigg|_{\mathbf{p},\mathbf{x},t} + \Delta t\,\frac{\partial f}{\partial t}\bigg|_{\mathbf{p},\mathbf{x},t}$$

$$-\cancel{f} = (\Delta f)_c$$

divide by dt, we have

$$\frac{\partial f}{\partial t} + \underbrace{\frac{\Delta\mathbf{x}}{dt}}_{\mathbf{v}}\cdot\frac{\partial f}{\partial \mathbf{x}} + \underbrace{\frac{\Delta\mathbf{p}}{\Delta t}}_{\mathbf{F}\ \leftarrow\ \text{force}}\cdot\frac{\partial f}{\partial \mathbf{p}} = \frac{(\Delta f)_c}{\Delta t}$$

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \mathbf{F}\cdot\nabla_p f = \left(\frac{\partial f}{\partial t}\right)_{\text{collisions}}}$$

Boltzmann eq.

if no collisions, we get the collisionless Boltzmann eq., or Vlasov eq.

$$\boxed{\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f + \underset{\sim}{F} \cdot \nabla_p f = 0}$$

can be written

$$\frac{Df}{Dt} = 0$$ "Liouville's theorem"

"f is constant along phase-space trajectories"

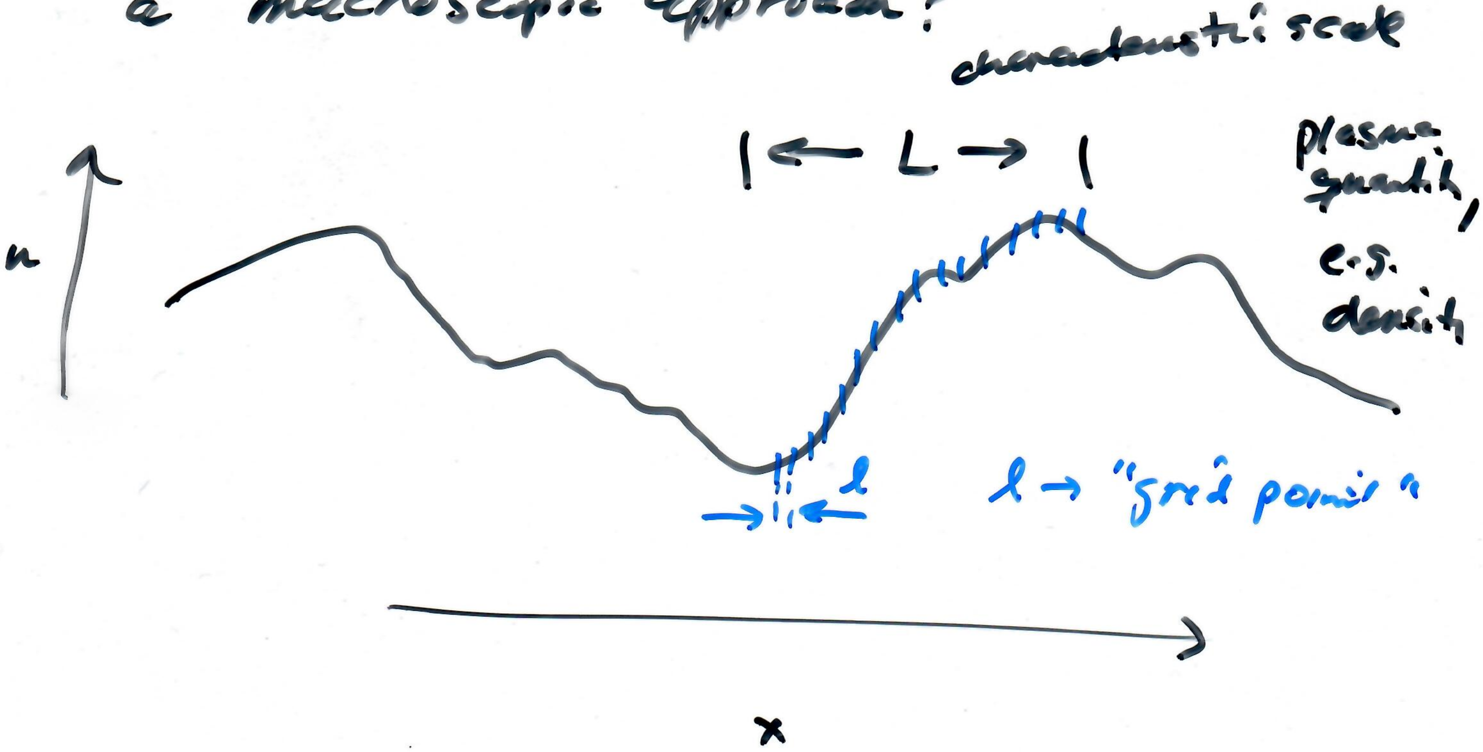
$$\underset{\sim}{F} = q \underset{\sim}{E} + \frac{q}{c} \underset{\sim}{v} \times \underset{\sim}{B}$$ for example

$$= \frac{G m M_\odot}{r^2} \hat{r}$$ gravity, etc.

by taking moments of Vlasov eq. we obtain macroscopic equations. (like hydrodynamics, MHD)

Macroscopic Eq.

~~A~~ preliminary consideration – under what conditions is it useful to ~~[illegible]~~ use a macroscopic approach?



require $l \ll L$

also require excellent statistics

$n l^3 \gg 1$ n → typical number density for $n\ell^3 = 10^{10}$ (5 decimal places accuracy) ~ 1000 grid pts.

	n	ℓ	L
solar corona	$10^7\ \text{cm}^{-3}$	10 cm	0.1 km
solar wind 1 AU	$5\ \text{cm}^{-3}$	10 m	10 km
Interstellar medium	$0.1\ \text{cm}^{-3}$	~ 50 m	50 km

we proceed to derive macroscopic eq's from a useful theorem

$$\frac{\partial}{\partial t} W + \nabla \cdot \underset{\sim}{W}_F = 0$$

$W \rightarrow$ conserved quantity

$\underset{\sim}{W}_F \rightarrow$ flux of conserved quantity