Macroscopic / Dynamical Equations

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$$\frac{\partial}{\partial t} W + \nabla \cdot \underset{\sim}{W}_F = 0$$

$W \rightarrow$ density of a conserved quantity

$\underset{\sim}{W}_F \rightarrow$ flux " " " "

Gauss's theorem, divergence theorem in calculus

Consider **mass** conservation

$$W = \text{mass density} = \frac{1}{V}\sum_i m_i$$

for like particles all m_i's are the same

$$\therefore \quad W = \frac{m}{V}\sum_i = \frac{Nm}{V}$$

N = # of particles in volume

$$= \rho = \text{mass density}$$

$$\underset{\sim}{W}_F = \frac{1}{V}\sum_i m_i \underset{\sim}{v}_i = \text{mass flux}$$

$$\underset{\sim}{v}_i = \underset{\sim}{u} + \underset{\sim}{w}_i$$

$\underset{\sim}{u} \rightarrow$ convective ~~[illegible]~~ speed

$$\langle \underset{\sim}{v}_i \rangle = \underset{\sim}{u}$$

$$\underset{\sim}{W}_F = \frac{m}{V}\sum_i (\underset{\sim}{u} + \cancel{\underset{\sim}{w}_i}^{\,0}) = \frac{m}{V}\underset{\sim}{u} N$$

$$= \rho \underset{\sim}{u}$$

therefore

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underset{\sim}{u}) = 0$$ mass cons. equation

<u>momentum conservation</u>

$$(W) = \frac{1}{V}\sum_i m_i \underset{\sim}{v}_i$$

$$= \rho \underset{\sim}{u}$$ momentum density

$$(\underset{\sim}{W}_F) = \frac{1}{V}\sum_i (m_i \underset{\sim}{v}_i)\underset{\sim}{v}_i$$ momentum flux

$$= \frac{m}{V}\sum_i (\underset{\sim}{u} + \underset{\sim}{w}_i)(\underset{\sim}{u} + \underset{\sim}{w}_i)$$

$$= \frac{m}{V}\sum_i (\underset{\sim}{u}\,\underset{\sim}{u} + \cancel{\underset{\sim}{w}_i\underset{\sim}{u}}^{\,0} + \cancel{\underset{\sim}{u}\,\underset{\sim}{w}_i}^{\,0} + \underset{\sim}{w}_i\underset{\sim}{w}_i)$$

$$= \frac{m}{V}\underset{\sim}{u}\,\underset{\sim}{u} N + \frac{m}{V}\sum \underset{\sim}{w}_i\underset{\sim}{w}_i \longrightarrow \underset{\sim}{P}$$ pressure tensor

$$= \rho \underset{\sim}{u} \underset{\sim}{u} + \underset{\approx}{P}$$

$$\therefore \quad \frac{\partial}{\partial t}(\rho \underset{\sim}{u}) + \nabla \cdot (\rho \underset{\sim}{u} \underset{\sim}{u} + \underset{\approx}{P}) = 0$$

this is for no external forces. But, if we have external forces, there will be sources of momentum (just add force per volume to RHS)

gravity $\underset{\sim}{F}^*_g = \frac{1}{V} \sum_i m_i \underset{\sim}{g}$ $\quad$ g = gravitational acceleration

$$= \rho \underset{\sim}{g}$$

elec. & mag. force $\underset{\sim}{F}^*_{EM} = \frac{1}{V} \sum_i \left(q_i \underset{\sim}{E} + \frac{q_i}{c} \underset{\sim}{v}_i \times \underset{\sim}{B} \right)$

$$= q n \underset{\sim}{E} + \frac{q}{c} n \underset{\sim}{u} \times \underset{\sim}{B} \qquad n = \frac{N}{V}$$

1368 W/m² → solar constant

radiation pressure $\underset{\sim}{F}^*_{R.P.} = \frac{1}{V} \sum_i \frac{Q_i F_0 A_i}{c} \left(\frac{1\,AU}{r} \right)^2 \hat{r}$

we have, for example

$$\frac{\partial}{\partial t}(\rho \underset{\sim}{u}) + \nabla \cdot (\rho \underset{\sim}{u}\underset{\sim}{u} + \underset{\approx}{P}) = \rho \underset{\sim}{g} + \frac{\rho}{m} q \underset{\sim}{E} + \frac{\rho}{m}\frac{q}{c} \underset{\sim}{u} \times \underset{\sim}{B}$$

This can also be written (using mass cons. eq.)

$$\underbrace{\rho \frac{\partial \underset{\sim}{u}}{\partial t} + \rho \underset{\sim}{u} \cdot \nabla \underset{\sim}{u}}_{\rho \frac{D\underset{\sim}{u}}{Dt}} = -\nabla \cdot \underset{\approx}{P} + \rho \underset{\sim}{g} + \frac{\rho}{m} q \underset{\sim}{E} + \frac{\rho}{m}\frac{q}{c} \underset{\sim}{u} \times \underset{\sim}{B}$$

$\rho \frac{D\underset{\sim}{u}}{Dt}$ where $\frac{D}{Dt} \to$ convective derivative

$$= \frac{\partial}{\partial t} + \underset{\sim}{u} \cdot \nabla$$

<u>Consider a different approach to deriving macroscopic equation</u>

start w/ Vlasov eq.

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \nabla f + \frac{\underset{\sim}{F}}{m} \cdot \nabla_v f = 0$$

we get macroscopic eq.'s by taking moments

1^{st} moment $\to$ multiply by m, integrate $d^3\underset{\sim}{v}$

2^{nd} moment $\to$ " " $m\underset{\sim}{v}$, " "

3^{rd} moment $\to$ " " mv^2, " "

define $\langle \phi \rangle = \frac{1}{n} \int \phi f \, d^3 \underset{\sim}{v}$

consider the function $\psi(\underset{\sim}{v})$. multiply Vlasov by ψ, integrate over $d^3 \underset{\sim}{v}$

$$\int d^3 \underset{\sim}{v} \; \psi(\underset{\sim}{v}) \left[\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \nabla f + \frac{\underset{\sim}{F}}{m} \cdot \nabla_{\underset{\sim}{v}} f \right] = 0$$

leads to

$$\frac{\partial}{\partial t}\left(n \langle \psi(\underset{\sim}{v}) \rangle\right) + \nabla \cdot \left(n \langle \psi \psi \rangle\right) = n \left\langle \frac{\underset{\sim}{F}}{m} \cdot \frac{\partial \psi}{\partial \underset{\sim}{v}} \right\rangle$$

1st moment $\quad \psi = m$

2nd moment $\quad \psi = m \underset{\sim}{v}$

3rd moment $\quad \psi = \frac{1}{2} m v^2$

we obtain the same equations as before

we find also: $\quad n = \int f \, d^3 \underset{\sim}{v}$

$$\underset{\sim}{u} = \frac{1}{n} \int f \underset{\sim}{v} \, d^3 \underset{\sim}{v}$$

$$P = m \int f \langle \underset{\sim}{w} \underset{\sim}{w} \rangle \, d^3 \underset{\sim}{v}$$

Consider the special case of a M-B distribution

$$f = \frac{n}{(2\pi m kT)^{3/2}} e^{-\frac{m(v_x^2+v_y^2+v_z^2)}{2kT}}$$

what is $\underset{\approx}{P}$?

$$P_{xx} = m\int_{-\infty}^{\infty} dv_z \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_x \frac{n}{(2\pi m kT)^{3/2}} e^{-\frac{m(v_x^2+v_y^2+v_z^2)}{2kT}} v_x v_x$$

$$\vdots$$

$$= nkT \equiv P_{yy} \equiv P_{zz}$$

$$P_{xy} = m\int_{-\infty}^{\infty} dv_z \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_x \frac{n}{(2\pi m kT)^{3/2}} e^{-\frac{m(v_x^2+v_y^2+v_z^2)}{2kT}} v_x v_y$$

$$= 0 \equiv P_{yx}, P_{zx}, P_{xz}, P_{yz}, P_{zy}$$

$$\underset{\approx}{P} = \begin{pmatrix} nkT & 0 & 0 \\ 0 & nkT & 0 \\ 0 & 0 & nkT \end{pmatrix} = nkT \underset{\approx}{I} = P \underset{\approx}{I}$$

in this case $\nabla \cdot \mathbf{P} = \nabla \cdot (P \mathbf{I}) = \nabla P$

we would have

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \rho \mathbf{g} + \frac{\rho}{m}\left(q \mathbf{E} + \frac{q}{c} \mathbf{u} \times \mathbf{B}\right)$$

MHD equations

in plasmas, we have multi-components

protons, electrons, neutrals, ions

recall

$$q^* = \text{charge density} = \sum_{i} \overset{\text{charge}}{q_i}\ \overset{\text{\# density}}{n_i}$$

i ← species

$$\mathbf{J} = \text{current density} = \sum_i q_i\, n_i\, \mathbf{u}_i \quad \leftarrow \text{bulk speed}$$

we write macroscopic eq's for each species, and combine using these definitions, and also define

(take a proton, electron, neutral plasma)

$$\rho = \text{~~center of mass~~ total mass density} = \rho_p + \rho_e + \rho_n$$

$$\underset{\sim}{U} = \text{center of mass velocity} = \frac{\rho_p \underset{\sim}{u}_p + \rho_e \underset{\sim}{u}_e + \rho_n \underset{\sim}{u}_n}{\rho}$$

$$\underset{\approx}{P} = \text{total pressure} = \underset{\approx}{P}_p + \underset{\approx}{P}_e + \underset{\approx}{P}_n$$

we obtain, for example:

$$\frac{\partial}{\partial t} \rho_p + \nabla \cdot (\rho_p \underset{\sim}{u}_p) = 0 \qquad \text{proton}$$

$$\frac{\partial}{\partial t} \rho_e + \nabla \cdot (\rho_e \underset{\sim}{u}_e) = 0 \qquad \text{elec.}$$

$$\frac{\partial}{\partial t} \rho_n + \nabla \cdot (\rho_n \underset{\sim}{u}_n) = 0 \qquad \text{neutral}$$

<u>and</u>

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{U}) = 0 \qquad \text{cons. of mass of plasma}$$

we also find

$$\frac{\partial}{\partial t}(\rho \underset{\sim}{\bar{U}}) + \nabla\cdot(\rho \underset{\sim}{\bar{U}}\underset{\sim}{\bar{U}}) = -\nabla\cdot\underset{\approx}{P} + q^* \underset{\sim}{E} + \frac{1}{c}\underset{\sim}{J}\times\underset{\sim}{B}$$

(no gravity)

MHD momentum eq.

or, scalar pressure, $n_i = n_e$ quasineutral we would get

$$\rho\frac{\partial \underset{\sim}{u}}{\partial t} + \rho\underset{\sim}{u}\cdot\nabla\underset{\sim}{u} = -\nabla p + \frac{1}{c}\underset{\sim}{J}\times\underset{\sim}{B}$$