Recall the MHD momentum equation

$$\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\nabla \cdot \underset{\approx}{P} + \overset{①}{q^* \underset{\sim}{E}} + \overset{②}{\frac{1}{c} \underset{\sim}{J} \times \underset{\sim}{B}}$$

ignore gravity
single fluid

$$q^* = e n_p - e n_e \qquad \text{(proton-electron plasma only)}$$

$$\underset{\sim}{J} = e n_p \underset{\sim}{u}_p - e n_e \underset{\sim}{u}_e$$

Look @ term ② first

$$\frac{1}{c} \underset{\sim}{J} \times \underset{\sim}{B} = \frac{1}{c}\left(\frac{c}{4\pi} \nabla \times \underset{\sim}{B}\right) \times \underset{\sim}{B}$$

Maxwell Ampere's law

$$\underset{\sim}{J} = \frac{c}{4\pi} \nabla \times \underset{\sim}{B} + \underbrace{\frac{1}{c} \frac{\partial \underset{\sim}{E}}{\partial t}}_{\substack{\text{this term} \\ \text{is } \mathcal{O}\left(\left(\frac{U}{c}\right)^2\right) \\ \times \text{ first term}}}$$

$$= \frac{1}{4\pi} (\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B} = -\frac{1}{4\pi} \underset{\sim}{B} \times (\nabla \times \underset{\sim}{B})$$

vector identity

$$= -\frac{1}{4\pi}\left(\nabla \frac{B^2}{2} - \underset{\sim}{B} \cdot \nabla \underset{\sim}{B}\right)$$

$$= -\nabla\left(\frac{B^2}{8\pi}\right) + \frac{1}{4\pi} \underset{\sim}{B} \cdot \nabla \underset{\sim}{B}$$

$$\therefore \quad \rho \frac{D\underset{\sim}{v}}{Dt} = -\nabla\cdot\left(\underset{\approx}{P} + \frac{B^2}{8\pi}\underset{\approx}{I}\right) + \frac{1}{4\pi}\underset{\sim}{B}\cdot\nabla\underset{\sim}{B} + q^*\underset{\sim}{E}$$

magnetic pressure force

magnetic tension force (per volume)

if pressure is isotropic, the first term on the RHS is

$$-\nabla\left(P + B^2/8\pi\right)$$

define plasma "beta", a fundamental plasma parameter

$$\beta = \frac{P}{(B^2/8\pi)} \quad \text{definition}$$

$$\beta_e = \text{electron beta} = \frac{n_e k T_e}{B^2/8\pi}$$

$$\beta_p = \text{proton beta} = \frac{n_p k T_p}{B^2/8\pi}$$

$\beta \gg 1$ photosphere

$\beta \ll 1$ corona

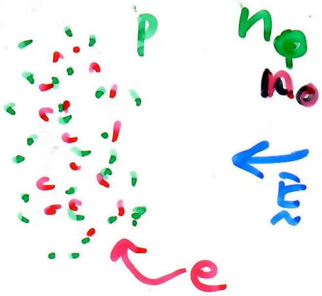
$\beta \sim 1$ at 1 AU in solar wind

$\beta \sim 1$ ISM

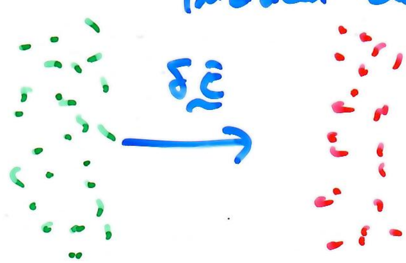
Now consider term 1

$$q^* \underline{E}$$

consider a proton-electron plasma, with protons & electrons having equal temperatures



some time later



~~Naturally~~ the distribution is M-B

$$f \sim e^{-E/kT} \approx e^{-\frac{q\phi}{kT}} \leftarrow \text{electric potential}$$

$E = +q\phi$ (kinetic energy!)

$$n_p = n_0 e^{-\frac{e\phi}{kT}}$$

$$n_e = n_0 e^{\frac{e\phi}{kT}}$$

$$\therefore \quad \rho^* = e n_p - e n_e$$

$$= e n_0 \left(e^{-\frac{e\phi}{kT}} - e^{+\frac{e\phi}{kT}} \right)$$

if $e\phi \ll kT$, we expand each to get

$$\rho^* \approx -e n_0 \left(2 \frac{e\phi}{kT} \right)$$

recall Poisson eq. $\nabla \cdot \underset{\sim}{E} = 4\pi \rho^*$

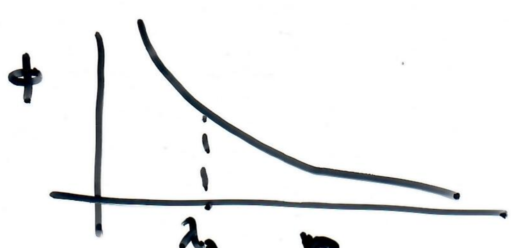
$$\Rightarrow \quad \nabla \cdot \underset{\sim}{E} = -8\pi e^2 n_0 \frac{\phi}{kT}$$

$$\underset{\sim}{E} = -\nabla \phi$$

$$\therefore \quad -\nabla^2 \phi = -8\pi e^2 n_0 \frac{\phi}{kT}$$

$$\Rightarrow \quad \left(\nabla^2 - \frac{1}{\lambda_D^2} \right) \phi = 0$$

where $\lambda_D = \left(\frac{kT}{8\pi e^2 n_0} \right)^{1/2}$ = Debye Length (a fundamental plasma parameter)



ϕ decays over a scale λ_D

$$\phi \sim \frac{1}{r} e^{-r/\lambda_D}$$

external field "shorted out" on scales $\gg \lambda_D$

~~Debye Length~~	n	T	λ_D	L
photosphere				
corona	$10^7\ \mathrm{cm}^{-3}$	$10^6\ \mathrm{K}$	$4\times10^{-6}\ \mathrm{cm}$	$\sim 5\ \mathrm{cm}$
solar wind at 1 AU	$5\ \mathrm{cm}^{-3}$	$10^5\ \mathrm{K}$	0.03 km	10 km
ISM	$0.1\ \mathrm{cm}^{-3}$	8000 K	.02 km	50 km

As a result, there are <u>no</u> charge-separation electric fields in MHD

Thus, there is <u>NO</u> $q^*\underset{\sim}{E}$ term in <u>single fluid</u> MHD equation

leaving

$$\boxed{\rho \frac{\partial \underset{\sim}{v}}{\partial t} + \rho\, \underset{\sim}{v} \cdot \nabla \underset{\sim}{v} = -\nabla\left(P + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi} \underset{\sim}{B} \cdot \nabla \underset{\sim}{B}}$$

Single-fluid MHD momentum equation

we also need an equation for $\underset{\sim}{B}$

Faraday's law is

$$\frac{\partial \underset{\sim}{B}}{\partial t} = -c \nabla \times \underset{\sim}{E}$$

transformation speed

we showed before $\underset{\sim}{E} = -\frac{1}{c} \underset{\sim}{w} \times \underset{\sim}{B}$

$\underset{\sim}{w} = \underset{\sim}{v}$ IN IDEAL MHD

what is the proper transformation speed?
which frame has no electric field?
one can argue that the electron ~~frame~~ frame is best

$$\underset{\sim}{w} = \underset{\sim}{u}_e$$

also, note that in the limit $m_e \to 0$ the electron bulk momentum eq. is

$m_e = 0$ (no inertia)

$$\cancelto{0}{m_e n_e \frac{\partial \underset{\sim}{u}_e}{\partial t}} + \cancelto{0}{m_e n_e \underset{\sim}{u}_e} \cdot \nabla \underset{\sim}{u}_e = -\nabla P_e - e n_e \underset{\sim}{E} - \frac{e}{c} n_e \underset{\sim}{u}_e \times \underset{\sim}{B}$$

$$\Rightarrow \quad \underset{\sim}{E} = -\frac{1}{c} \underset{\sim}{u}_e \times \underset{\sim}{B} - \frac{1}{e n_e} \nabla P_e$$

take this as the $\underset{\sim}{E}$, and we have

$$\frac{\partial \underset{\sim}{B}}{\partial t} = -c \nabla \times \left[-\frac{1}{c} \underset{\sim}{u}_e \times \underset{\sim}{B} - \frac{1}{e n_e} \nabla P_e \right]$$

because we are still working on scales $\gg \lambda_D$

$\Rightarrow n_e \simeq n_p$ (quasi-neutrality)

$\therefore \quad n_e = n_p = n$

$$\Rightarrow \quad \frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{u}_e \times \underset{\sim}{B}) + \nabla \times \left(\frac{c}{en_e} \nabla P_e\right)$$

recall $\quad \underset{\sim}{J} = en_p \underset{\sim}{u}_p - en_e \underset{\sim}{u}_e$

$$= en(\underset{\sim}{u}_p - \underset{\sim}{u}_e)$$

and $\quad \underset{\sim}{U} = \dfrac{m_p n_p \underset{\sim}{u}_p + \cancel{m_e n_e}\, \underset{\sim}{u}_e}{m_p n_p + \cancel{m_e n_e}} \simeq \underset{\sim}{u}_p$

$$\therefore \quad \underset{\sim}{J} = en(\underset{\sim}{U} - \underset{\sim}{u}_e) \Rightarrow \underset{\sim}{u}_e = \underset{\sim}{U} - \frac{1}{ne} \underset{\sim}{J}$$

$$\therefore \quad \frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B}) + \nabla \times \left(-\frac{1}{ne} \underset{\sim}{J} \times \underset{\sim}{B} + \frac{c}{en} \nabla P_e\right)$$

$$\underset{\sim}{E} = -\frac{1}{c} \underset{\sim}{U} \times \underset{\sim}{B} + \frac{1}{nec} \underset{\sim}{J} \times \underset{\sim}{B} + \frac{1}{ne} \nabla P_e$$

HALL TERM

POLARIZATION ELECTRIC FIELD

HALL TERM ONLY IMPORTANT FOR scales

$\gg c/\omega_p$ (ION - inertial length)

$$\omega_p = \left(\frac{4\pi n e^2}{m_p}\right)^{1/2}$$ proton plasma frequency

POLARIZATION $\underset{\sim}{E}$ FIELD ONLY IMPORTANT on SCALES

$\gg$ thermal ion gyroradius

therefore

$$\underset{\sim}{E} \simeq -\frac{1}{c}\,\underset{\sim}{U} \times \underset{\sim}{B}$$ in most cases

(not magnetic reconnection, or shocks, plasma waves, etc.)

leaves

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B})$$ magnetic induction equation w/o resistivity

if there is finite electrical conductivity, there is a $\frac{1}{\sigma}\underset{\sim}{J}$ added to the electric field

("ohms" law)

$$\underset{\sim}{E} = -\frac{1}{c}\,\underset{\sim}{U}\times\underset{\sim}{B} + \frac{1}{nec}\,\underset{\sim}{J}\times\underset{\sim}{B} - \frac{1}{en_e}\nabla P_e + \frac{1}{\sigma}\,\underset{\sim}{J}$$

A form of the "Generalised" Ohm's law

scalar electrical conductivity (seldom justified!)