PTYS 558 2/12/2020

Magnetic Induction Eq. (cont.)

if we ignore Hall term & Polarization $\underset{\sim}{E}$ field, we have

$$\underset{\sim}{E} = -\frac{1}{c}\underset{\sim}{U} \times \underset{\sim}{B} + \frac{1}{\sigma}\underset{\sim}{J}$$

then, Faraday's law gives

$$\frac{\partial \underset{\sim}{B}}{\partial t} = -c\,\nabla \times \underset{\sim}{E}$$

$$= -c\,\nabla \times \left(-\frac{1}{c}\underset{\sim}{U} \times \underset{\sim}{B} + \frac{1}{\sigma}\underset{\sim}{J}\right)$$

$$= \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B}) - \frac{c}{\sigma}\nabla \times \underset{\sim}{J}$$

) Ampere's law

$$= \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B}) - \frac{c^2}{4\pi\sigma}\nabla \times (\nabla \times \underset{\sim}{B})$$

) vec. identity

$$= \underbrace{\nabla \times (\underset{\sim}{U} \times \underset{\sim}{B})}_{\text{advection term}} + \underbrace{\overbrace{\frac{c^2}{4\pi\sigma}}^{\to \eta}\nabla^2 \underset{\sim}{B}}_{\text{diffusion term}}$$

define Magnetic Reynolds # $\underbrace{}_{R_M}$ $= \dfrac{\text{advection term}}{\text{diffusion term}} = \dfrac{\frac{1}{L}UB}{\frac{c^2}{4\pi\sigma}\frac{B}{L^2}}$

$$R_M = \frac{LU\,4\pi\sigma}{c^2} = \frac{LU}{\eta} \quad ; \eta = \frac{c^2}{4\pi\sigma} = \text{mag. diffusivity}$$

what is R_μ in this room?

$$R_M = \frac{LU\, 4\pi\sigma}{c^2}$$

$L \simeq 1\,m \simeq 100\,cm$

$U \simeq 10\,mph \simeq 450\,\frac{cm}{s}$

$\sigma_{air} \simeq 4.5\times10^{-5}\,1/s$ (c.g.s.)

$$= \frac{(100)(450)(4\pi)(4.5\times10^{-5})}{(3\times10^{10})^2}$$

$$= 3\times10^{-20} \ll 1$$

note also, the plasma β in this room is

$$\beta = \frac{P}{(B^2/8\pi)} = \frac{1\text{ bar}}{B^2/8\pi} = \frac{10^6\,\frac{dyne}{cm^2}}{\frac{(1)^2}{8\pi}\,\frac{dyne}{cm^2}}$$

$$\simeq 8\pi\times10^6 \gg 1$$

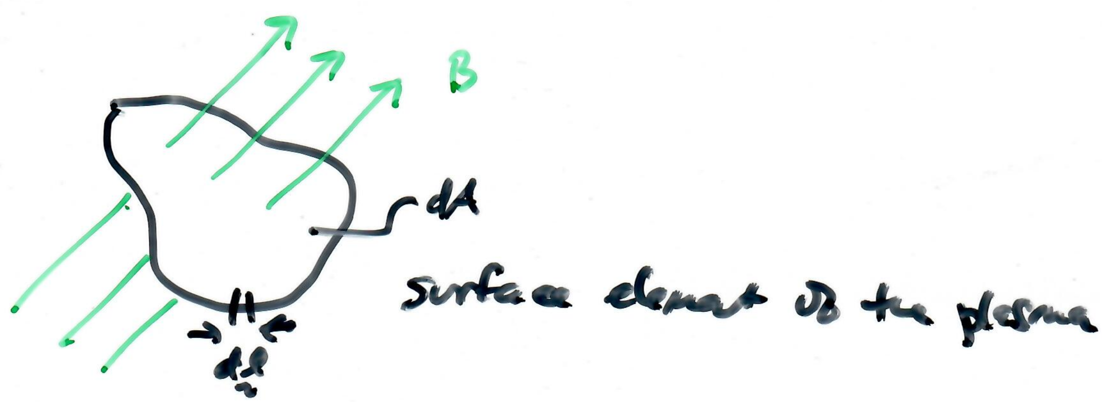
But, in most Astrophysical | Heliospheric plasmas

$R_m \gg 1$

Frozen-flux theorem

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{u} \times \underset{\sim}{B}) + \eta \nabla^2 \underset{\sim}{B}$$

if $\eta \to 0$, $\underset{\sim}{B}$ is "frozen" into the plasma flow



flux through area is $\int \underset{\sim}{B} \cdot d\underset{\sim}{A}$

$$\frac{\partial}{\partial t}\left(\int \underset{\sim}{B} \cdot d\underset{\sim}{A}\right) = \int \frac{\partial \underset{\sim}{B}}{\partial t} \cdot d\underset{\sim}{A} + \int \underset{\sim}{B} \cdot \frac{d\underset{\sim}{A}}{dt}$$

Leibniz theorem

$$= \int \frac{\partial B}{\partial t} \cdot d\underset{\sim}{A} + \oint \underset{\sim}{B} \cdot (\underset{\sim}{u} \times d\underset{\sim}{l})$$

vec identity

$$= \int \frac{\partial B}{\partial t} \cdot d\underset{\sim}{A} + \oint (\underset{\sim}{B} \times \underset{\sim}{u}) \cdot d\underset{\sim}{l}$$

$$= \int \frac{\partial B}{\partial t} \cdot d\underset{\sim}{A} - \oint (\underset{\sim}{u} \times \underset{\sim}{B}) \cdot d\underset{\sim}{l}$$

$\downarrow$ Stokes theorem

$$= \int \frac{\partial \underset{\sim}{B}}{\partial t} \cdot d\underset{\sim}{A} - \int \left[\nabla \times (\underset{\sim}{u} \times \underset{\sim}{B})\right] \cdot d\underset{\sim}{A}$$

$$= \int \left[\frac{\partial \underset{\sim}{B}}{\partial t} - \nabla \times (\underset{\sim}{u} \times \underset{\sim}{B})\right] \cdot d\underset{\sim}{A}$$

$$= 0 \quad \text{if} \quad \eta = 0$$

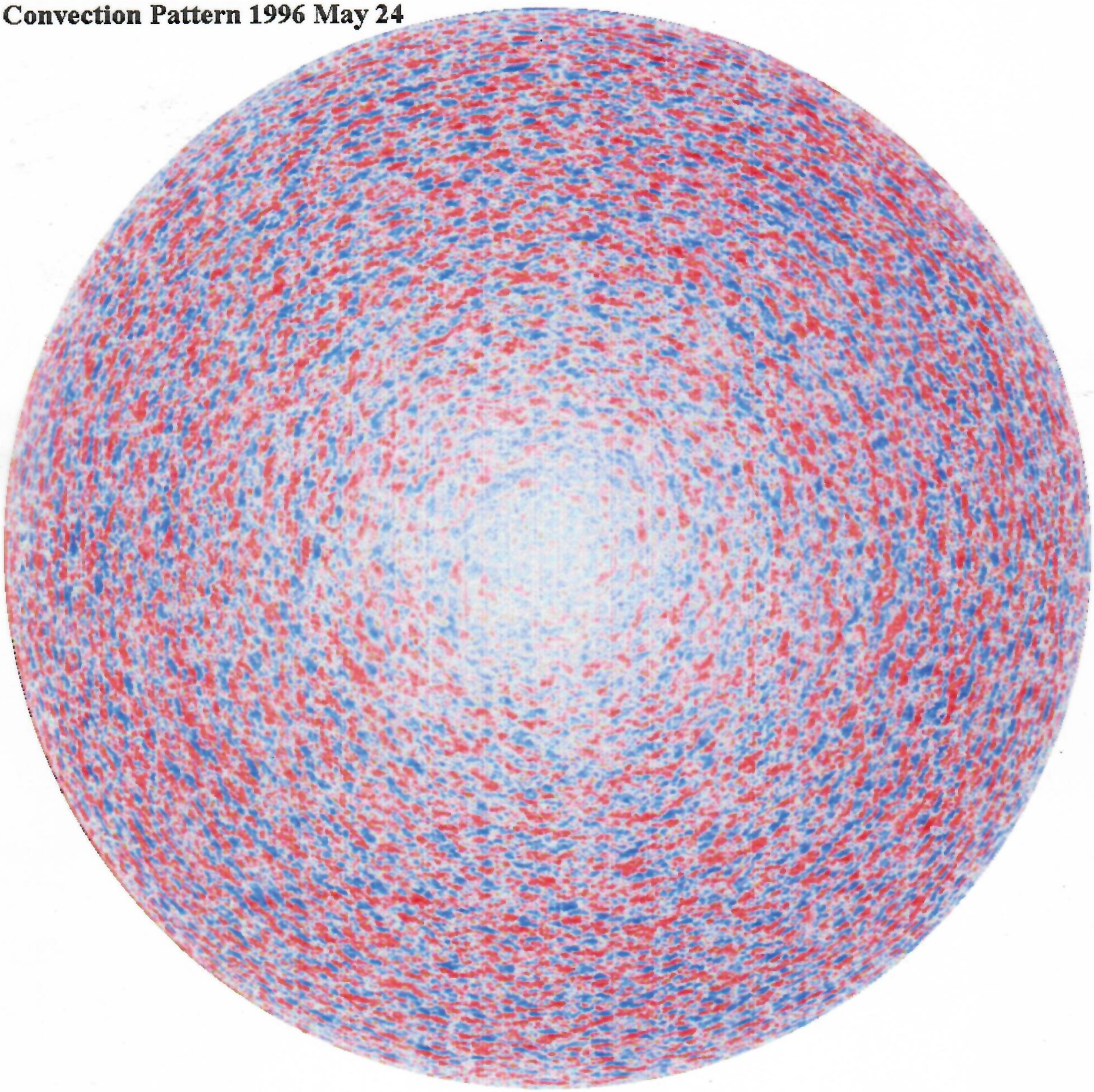
the $\nabla \times (\underset{\sim}{u} \times \underset{\sim}{B})$ term in the mag. induction eq. leads to <u>field amplification</u>

turbulent dynamo $\rightarrow$ strong field amplification

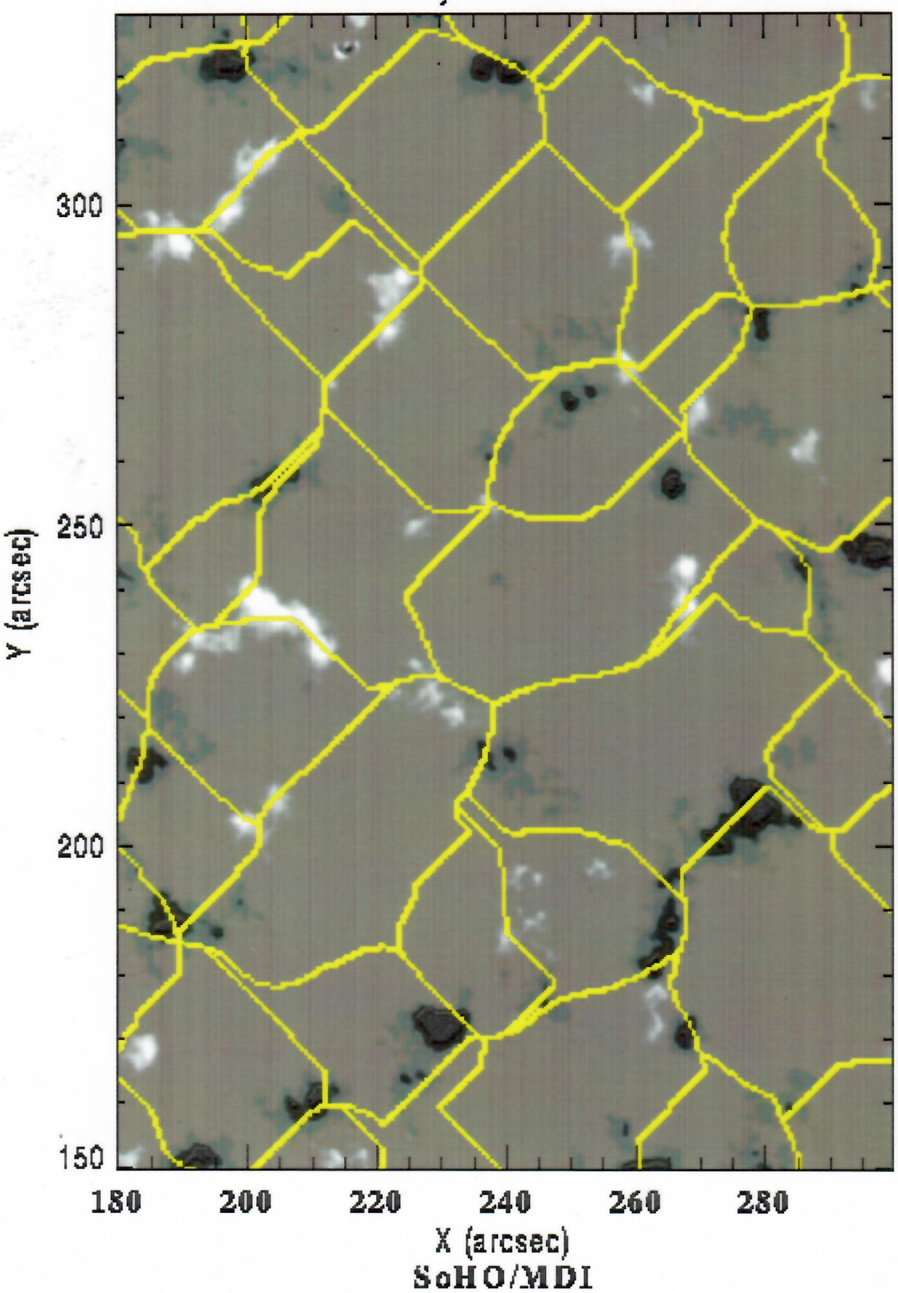
$\alpha - \omega$ dynamo $\rightarrow$ generates stellar & planetary fields

an example of a turbulent dynamo is the sun's "network" magnetic field

Convection Pattern 1996 May 24



23 Feb. 1996,16:44 to 21:03 UT

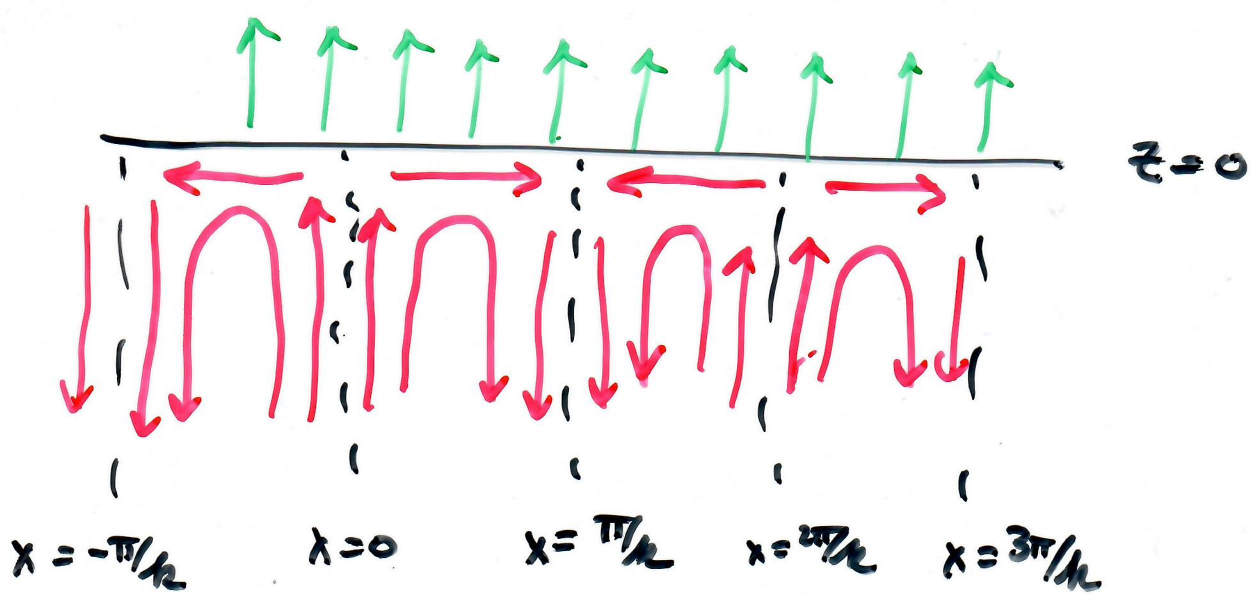


Consider the following kinematic representation of the transverse plasma flow at the Sun due to supergranulation

$\underset{\sim}{u} = (u_x, 0, u_z)$ $\underset{\sim}{B} = (0, 0, B(x))$

$u_x = u_0 \sin kx$

$u_z = -kz \cos kx$



initially $\frac{B^2}{8\pi} << \rho u^2$ ρ = plasma density

How does $B(x,t)$ evolve with x & t ?

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{u} \times \underset{\sim}{B})$$

Ignore mag. diffusion

$$\underset{\sim}{B} = B(x,t)\,\hat{z}$$

u_x, B do not depend on z

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \cancelto{0}{\frac{\partial}{\partial z}(u_x B)}\,\hat{x} - \frac{\partial}{\partial x}(u_x B)\,\hat{z}$$

$$\Rightarrow \quad \frac{\partial B(x,t)}{\partial t} = -\frac{\partial}{\partial x}(u_x B)$$