Kinematic dynamo – example (supergranular turbulent dynamo):

recall from last time, we had

$$\frac{\partial B}{\partial t} = -\frac{\partial}{\partial x}(u_x B)$$

multiply both sides by u_x, and note $u_x = u_x(t)$

$$\Rightarrow \quad \frac{\partial}{\partial t}(u_x B) + u_x \frac{\partial}{\partial x}(u_x B) = 0$$

$$\frac{D}{Dt}(u_x B) = 0 \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + u_x \frac{\partial}{\partial x}$$

this means $u_x B$ = constant along flow streamlines

what is the streamline?

streamline eq. is

$$\frac{dx}{dt} = u_x = u_0 \sin kx$$

$$\Rightarrow \int_{x_0}^{x} dx/\sin kx = \int_{t_0}^{t} u_0 \, dt$$

$$\Rightarrow \quad -\frac{1}{k}\ln\left|\csc kx + \cot kx\right|\Big]_{x_0}^{x} = u_0(t-t_0)$$

$$\vdots$$

$$\Rightarrow \quad -\frac{1}{k}\ln\left|\frac{\cot\left(\frac{kx}{2}\right)}{\cot\left(\frac{kx_0}{2}\right)}\right| = u_0(t-t_0)$$

we choose $t_0 = 0$, giving

$$-ku_0 t = \ln\left|\frac{\cot\left(\frac{kx}{2}\right)}{\cot\left(\frac{kx_0}{2}\right)}\right|$$

this leads to. After lots of tedious algebra

$$\sin kx_0 = \frac{2\tan\left(\frac{kx}{2}\right)e^{-ku_0 t}}{1+\tan^2\left(\frac{kx}{2}\right)e^{-2ku_0 t}}$$

Since $u_x B$ is constant along a streamline, we have

$$u_x(x)\,B(x,t) = \underbrace{u_x(x_0)}_{u_0 \sin kx_0}\ \underbrace{B(x_0,t_0)}_{B_0}$$

$$\therefore \quad B(x,t) = B_0 \frac{u_0 \sin kx_0}{u_0 \sin kx}$$

$$\boxed{B(x,t) = B_0 \frac{1}{\sin kx} \frac{2 \tan\left(\frac{kx}{2}\right) e^{-ku_0 t}}{1 + \tan^2\left(kx/2\right) e^{-2ku_0 t}}}$$

@ $x = 0$ (updraft)

$$B(0,t) = B_0 e^{-ku_0 t}$$

B weakens

@ $x = \pi/k$ (downdraft)

$$B(0,t) = B_0 e^{ku_0 t}$$

B strengthens

τ = growth ~~scale~~ timescale at downdrafts $\sim \frac{1}{ku_0}$

$$\sim \frac{1}{\left(\frac{2\pi}{L_{sg}}\right) V_{sg}}$$

$L_{sg} \simeq 30$ Mm

$V_{sg} \simeq 1/2$ km/s

(observations)

$$\tau \simeq \frac{L_{sg}}{2\pi V_{sg}} \simeq \frac{3\times10^{9}\ \text{cm}}{(2\pi)(5\times10^{4}\ \frac{\text{cm}}{\text{s}})}$$

$\simeq$ 2.6 hrs

A supergranule "lives" for a day or two

$\therefore$ there are ~ 8 growth time scales in the life of a supergranule

$\therefore$ field grows by a factor $e^8 \sim$ thousands

this agrees w/ observation

note also that for a 1000 Gauss mag. field in solar photosphere, has less pressure (but comparable to) thermal pressure

e.g. $\frac{B^2}{8\pi} \sim \frac{(1000)^2}{8\pi} \sim 4\times10^{4}\ \frac{\text{dyne}}{\text{cm}^2}$

$$(nkT)_{\text{photosphere}} \simeq (2\times10^{17})(1.4\times10^{-16})(5800)\ \frac{\text{dyne}}{\text{cm}^2}$$

$$\simeq 1.6\times10^{5}\ \frac{\text{dyne}}{\text{cm}^2}$$

~~How to~~

Return to MHD equations.

The energy equation

recall, $\frac{\partial}{\partial t} w + \nabla \cdot \underset{\sim}{w} = 0$

$w \rightarrow$ density conserved quantity

$\underset{\sim}{w} \rightarrow$ flux of conserved quantity

<u>energy</u>

$$w = \frac{1}{V} \sum_i \frac{1}{2} m_i (\underset{\sim}{v}_i \cdot \underset{\sim}{v}_i)$$

$$\underset{\sim}{v}_i = \underset{\sim}{u} + \underset{\sim}{w}_i$$

$$= \frac{1}{V} \sum_i \frac{1}{2} m_i (\underset{\sim}{u} + \underset{\sim}{w}_i) \cdot (\underset{\sim}{u} + \underset{\sim}{w}_i)$$

$$= \frac{m}{2V} \sum_i \left(u^2 + \overbrace{(\underset{\sim}{w}_i \cdot \underset{\sim}{u})}^{\rightarrow 0} + \overbrace{(\underset{\sim}{u} \cdot \underset{\sim}{w}_i)}^{\rightarrow 0} + w_i^2\right)$$

$$= \frac{1}{2} \rho u^2 + \frac{1}{2} P_{ii}$$

$P_{ii} \rightarrow$ trace of $\underset{\approx}{P}$

$= P_{xx} + P_{yy} + P_{zz}$

$$\mathbf{W}_F = \frac{1}{V} \sum_i \tfrac{1}{2} m_i (\mathbf{v}_i \cdot \mathbf{v}_i)\, \mathbf{v}_i$$

$$= \frac{m}{2V} \sum_i \left[(\mathbf{u} + \mathbf{w}_i) \cdot (\mathbf{u} + \mathbf{w}_i) \right] (\mathbf{u} + \mathbf{w}_i)$$

$$= \frac{m}{2V} \sum_i \Big[u^2 \mathbf{u} + \cancelto{0}{u^2 \mathbf{w}_i} + \cancelto{0}{(2 \mathbf{w}_i \cdot \mathbf{u})}\, \mathbf{u}$$

$$+ (2 \mathbf{w}_i \cdot \mathbf{u})\, \mathbf{w}_i + w_i^2 \mathbf{u} + w_i^2 \mathbf{w}_i \Big]$$

$$= \tfrac{1}{2} \rho u^2 \mathbf{u} + \mathbf{P} \cdot \mathbf{u} + \tfrac{1}{2} P_{ii} \mathbf{u} + \mathbf{q}$$

where $\mathbf{q}$ = heat flux vector $= \dfrac{m}{2V} \sum_i w_i^2 \mathbf{w}_i$

Thus, we have

$$\frac{\partial}{\partial t} \left(\tfrac{1}{2} \rho u^2 + \tfrac{1}{2} P_{ii} \right) + \nabla \cdot \Big[\left(\tfrac{1}{2} \rho u^2 + \tfrac{1}{2} P_{ii} \right) \mathbf{u} + \mathbf{P} \cdot \mathbf{u} + \mathbf{q} \Big] = 0$$

↑ in the absence of work done by external forces if $\underset{\sim}{\Phi} = 0$, then it can be shown

$$\frac{D}{Dt}\left(P/\rho^{\gamma}\right) = 0$$

γ = ratio of specific heats
= 5/3 monatomic gas

also P is isotropic

$\Rightarrow$ P/ρ^{γ} = constant along streamlines of flow

adiabatic ~~gas~~ gas

if there is an external force from $\underset{\sim}{E}$
($\underset{\sim}{B}$ does no work on charged particles)
then the RHS of the energy equation
would be

$$\frac{1}{V}\sum_i q_i \underset{\sim}{E}\cdot\underset{\sim}{v}_i = \frac{1}{V} q \sum_i \underset{\sim}{E}\cdot(\underset{\sim}{u}+\underset{\sim}{w}_i)$$

$$= n q \underset{\sim}{E}\cdot\underset{\sim}{u}$$

writing energy eq. for protons & electrons
and then adding, leads to the RHS being

$$= \underset{\sim}{E}\cdot\underset{\sim}{J}$$

in non-relativistic plasma flows this is

$$= \underset{\sim}{E}\cdot\left(\frac{c}{4\pi}\nabla\times\underset{\sim}{B}\right)$$

vector identity

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\Rightarrow \nabla\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B})$$

$$\cancel{= \mathbf{B}\cdot(\nabla\times\mathbf{E})}$$

$$\Rightarrow \mathbf{E}\cdot(\nabla\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \nabla\cdot(\mathbf{E}\times\mathbf{B})$$

$$= \mathbf{B}\cdot\left(-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right) - \nabla\cdot(\mathbf{E}\times\mathbf{B})$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{2}B^2\right) - \nabla\cdot(\mathbf{E}\times\mathbf{B})$$

$$\therefore \mathbf{E}\cdot\mathbf{J} = \frac{c}{4\pi}\mathbf{E}\cdot(\nabla\times\mathbf{B})$$

$$= -\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) - \nabla\cdot\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{B}\right)$$

mag. pressure — Poynting flux

The complete MHD energy equation (monatomic gas, isotropic pressure) is often written

$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \frac{1}{\gamma-1} + \frac{B^2}{8\pi}\right) + \nabla\cdot\left[\left(\frac{1}{2}\rho u^2 + \frac{\gamma}{\gamma-1}p\right)\underset{\sim}{u} + \frac{c}{4\pi}\underset{\sim}{E}\times\underset{\sim}{B} + \underset{\sim}{Q}\right] = 0}$$

can include work done by gravity, e.g. here

$\gamma = 5/3$

p isotropic

In summary the ideal MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{u}) = 0$$

$$\rho \left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} \right) = -\nabla P + \frac{1}{c} \underset{\sim}{J} \times \underline{B} - \rho \underset{\sim}{g}$$

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{u} \cdot \nabla \right) \left(P \rho^{-5/3} \right) = 0$$

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{u} \times \underset{\sim}{B})$$

auxillary eq!s

$$\nabla \cdot \underline{B} = 0$$

$$\underset{\sim}{J} = \frac{c}{4\pi} \nabla \times \underset{\sim}{B}$$

$$\underset{\sim}{E} = -\frac{1}{c} \underset{\sim}{u} \times \underset{\sim}{B}$$