Grad-Shafranov equation

An equation of reduced MHD that describes an equilibrium ~~potential~~ two-dimensional magnetic field.

Consider a 2-D magnetic field of the form

$$\underset{\sim}{B} = \left[B_x(x,y),\ B_y(x,y),\ B_z(x,y)\right]$$

this is sometimes called a 2-½-D mag. field (2 spatial dimensions, 3 components of vector)

note that $\nabla \cdot \underset{\sim}{B} = 0$ implies

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$$

We now define a function, A, similar to a stream function in fluid mechanics, such that

$$B_x = \frac{\partial A(x,y)}{\partial y} \qquad (*)$$

$$B_y = -\frac{\partial A(x,y)}{\partial x}$$

Note that this is a solution to $\nabla \cdot \underset{\sim}{B} = 0$

We also note that

$$\nabla \times (A\hat{z}) = \frac{\partial A}{\partial y}\hat{x} - \frac{\partial A}{\partial x}\hat{y} = B_x\hat{x} + B_y\hat{y}$$

$\Rightarrow$ A is the z component of the vector potential

Furthermore, we have

$$\underset{\sim}{B} \cdot \nabla A = B_x \frac{\partial A}{\partial x} + B_y \frac{\partial A}{\partial y}$$

$$= -B_x B_y + B_y B_x = 0$$

$\Rightarrow$ A = constant along field lines

$\therefore$ A defines field lines

Recall the MHD mom. equation in static equil. (ignoring gravity)

$$\nabla p - \frac{1}{c}\,\underset{\sim}{J} \times \underset{\sim}{B} = 0$$

Recall from class notes of 9/27/07 page 8

$$\frac{1}{c}\,\underset{\sim}{J} \times \underset{\sim}{B} = -\nabla\left(B^2/8\pi\right) + \frac{1}{4\pi}\left(\underset{\sim}{B}\cdot\nabla\right)\underset{\sim}{B}$$

Look at x-component of mom. eq.

$$\frac{\partial}{\partial x}\left(p + \frac{B_x^2 + B_y^2 + B_z^2}{8\pi}\right) - \frac{1}{4\pi} B_x \frac{\partial B_x}{\partial x} - \frac{1}{4\pi} B_y \frac{\partial B_x}{\partial y} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial x}\left(p + \frac{B_z^2}{8\pi}\right) + \frac{1}{4\pi} B_y \frac{\partial B_y}{\partial x} - \frac{1}{4\pi} B_y \frac{\partial B_x}{\partial y} = 0$$

substitute from above (*)

$$\left.\begin{aligned} &\frac{\partial}{\partial x}\left(p + \frac{B_z^2}{8\pi}\right) - \frac{B_y}{4\pi}\nabla^2 A = 0 \\ &\text{similarly the y-component gives} \\ &\frac{\partial}{\partial y}\left(p + \frac{B_z^2}{8\pi}\right) + \frac{B_x}{4\pi}\nabla^2 A = 0 \end{aligned}\right\} \quad (**)$$

multiply top eq. by B_x and the bottom by B_y, to give (and add)

$$\underset{\sim}{B} \cdot \nabla \left(p + B_z^2/8\pi \right) = 0$$

$$\Rightarrow \quad p + B_z^2/8\pi = \text{constant along magnetic field lines}$$

Also, take $\underset{\sim}{B} \cdot$ [momentum eq.] to find

$$\underset{\sim}{B} \cdot \nabla p = 0$$

$$\Rightarrow \quad p = \text{constant along field lines}$$

$$\Rightarrow \quad B_z = \text{" \quad " \quad "}$$

Because A = const. along field lines, the p & B_z are both functions of A

Define $\mathbb{P} = p + B_z^2/8\pi$

$$\frac{\partial \mathbb{P}}{\partial x} = \frac{d\mathbb{P}}{dA}\frac{\partial A}{\partial x} \quad ; \quad \frac{\partial \mathbb{P}}{\partial y} = \frac{d\mathbb{P}}{dA}\frac{\partial A}{\partial y}$$

$$= -B_y \frac{d\mathbb{P}}{dA} \qquad\qquad = B_x \frac{d\mathbb{P}}{dA}$$

Use these and combine with (**) to give

$$\frac{d}{dA}\left(p + \frac{B_z^2}{8\pi}\right) + \frac{1}{4\pi}\nabla^2 A = 0$$

$$\Rightarrow \quad \boxed{\nabla^2 A + 4\pi \frac{d}{dA}\left(p + \frac{B_z^2}{8\pi}\right) = 0}$$

Grad-Shafranov eq.

Note the similarity to Poisson's eq. with $\frac{d}{dA}(p + B_z^2/8\pi)$ being the "charge density" and A being the "~~scalar~~ electrostatic potential"

The current density is

$$\underset{\sim}{J} = \frac{c}{4\pi} \nabla \times \underset{\sim}{B} = \frac{c}{4\pi}\left(\frac{\partial B_z}{\partial y}, -\frac{\partial B_z}{\partial x}, -\nabla^2 A\right)$$

$$\vdots$$

$$\boxed{\underset{\sim}{J} = \frac{c}{4\pi}\left(\underset{\sim}{B}\frac{dB_z}{dA} + 4\pi \frac{dp}{dA}\hat{z}\right)}$$

$\therefore$ $p =$ constant gives a force-free field ($\underset{\sim}{J} \parallel \underset{\sim}{B}$) and $\frac{dB_z}{dA}$ is the constant α

Also, for a force-free field, Grad-Shafranov becomes

$$\nabla^2 A + \frac{d}{dA}\left(\frac{1}{2} B_z^2\right) = 0$$

Many solar physics problems start with this <u>or</u> test their assumed model for consistency with this