Calculation of Cell Areas on an Oblate Planet

Purpose: To calculate the cell areas on an oblate spheroidal planet. A "cell" is defined as a region bounded by two meridians of longitude and two meridians of latitude. It is approximately a rectangular patch (except for cells near the two poles, which are spherical triangles). To better approximation, a cell is a trapezoid. Here, we rigorously derive the area.

We define f as the flattening of the sphere,

$$f = \frac{a-b}{a} \tag{1}$$

where a and b are the equatorial and polar radii, respectively. Note $a > b$ for an oblate spheroid. For Earth, $f = 0.00335364$; for Mars, $f = 0.00647630$.

We begin with the definition of surface area in polar coordinates,

$$\text{Area} = \int_{\lambda_1}^{\lambda_2} \int_{\beta_1}^{\beta_2} r^2 \cos\beta \, d\beta \, d\lambda \tag{2}$$

where λ and β represent the longitude and latitude, respectively. For the sake of compactness, we do not carry the limits on the integrals in subsequent equations. However, the definite integrals remain to be evaluated after the dust settles.

The figure of an oblate spheroid is given (to order f^2) by,

$$r = a\Big(1 - f\sin^2\beta - \frac{3}{8}f^2\sin^2 2\beta\Big) \tag{3}$$

Substituting (3) into (2) and doing the integration in the λ direction gives:

$$\text{Area} = a^2\Delta\lambda \int \cos\beta\Big(1 - f\sin^2\beta - \frac{3}{8}f^2\sin^2 2\beta\Big)\Big(1 - f\sin^2\beta - \frac{3}{8}f^2\sin^2 2\beta\Big)\, d\beta \tag{4}$$

$$= a^2\Delta\lambda \int \cos\beta\Big[1 - f\sin^2\beta - \frac{3}{8}f^2\sin^2 2\beta + f^2\sin^4\beta + \frac{3}{8}f^3\sin^2\beta\sin^2 2\beta - \frac{3}{8}f^2\sin^2 2\beta + \frac{3}{8}f^3\sin^2 2\beta\sin^2 2\beta + \frac{9}{64}f^4\sin^4 2\beta\Big]\, d\beta \tag{5}$$

$$= a^2\Delta\lambda \int \Big[\cos\beta - f\sin^2\beta\cos\beta - \frac{3}{8}f^2\sin^2 2\beta\cos\beta + f^2\sin^4\beta\cos\beta + \frac{3}{8}f^3\sin^2\beta\sin^2 2\beta\cos\beta - \frac{3}{8}f^2\sin^2 2\beta\cos\beta + \frac{3}{8}f^3\sin^2 2\beta\sin^2 2\beta\cos\beta + \frac{9}{64}f^4\sin^4 2\beta\cos\beta\Big]\, d\beta \tag{6}$$

$$= a^2\Delta\lambda \int \Big[\cos\beta - f\sin^2\beta\cos\beta - \frac{3}{4}f^2\sin^2 2\beta\cos\beta + f^2\sin^4\beta\cos\beta + \frac{3}{4}f^3\sin^2\beta\sin^2 2\beta\cos\beta + \frac{9}{64}f^4\sin^4 2\beta\cos\beta\Big]\, d\beta \tag{7}$$

$$= a^2\Delta\lambda\Bigg[\sin\beta - \frac{f}{3}\sin^3\beta - \frac{3}{4}f^2\int\sin^2 2\beta\cos\beta\ d\beta + \frac{f^2}{5}\sin^5\beta$$

$$+ \frac{3}{4}f^3\int\sin^2\beta\sin^2 2\beta\cos\beta\ d\beta + \frac{9}{64}f^4\int\sin^4 2\beta\cos\beta\Bigg]\ d\beta \qquad (8)$$

$$= a^2\Delta\lambda\Bigg[\overbrace{\left(\sin\beta - \frac{f}{3}\sin^3\beta + \frac{f^2}{5}\sin^5\beta\right)\Bigg|_{\beta_1}^{\beta_2}}^{I} - \overbrace{\frac{3}{4}f^2\int\sin^2 2\beta\cos\beta\ d\beta}^{II}$$

$$+ \underbrace{\frac{3}{4}f^3\int\sin^2\beta\sin^2 2\beta\cos\beta\ d\beta\, II}_{I} + \underbrace{\frac{9}{64}f^4\int\sin^4 2\beta\cos\beta\ d\beta}_{IV}\Bigg] \qquad (9)$$

To do the integrals in terms *II–IV*, note the following trigonometric identities:

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ \sin^2 2\theta &= 4\cos^2\theta\cos^4\theta = 4(\sin^2\theta - \sin^4\theta)\\ \sin^4 2\theta &= 16(\cos^4\theta - 2\cos^6\theta + \cos^8\theta) = 16(\sin^4\theta - 2\sin^6\theta + \sin^8\theta)\end{aligned} \qquad (10a - 10c)$$

Term I is self-evident. Examining terms II–IV in turn,

II:

$$-\frac{3}{4}f^2\int\left(16\sin^4 - 2\beta\sin^6\beta + \sin^8\beta\right)\cos\beta\ d\beta$$

$$= -\frac{3}{4}f^2\left[\frac{16}{5}\sin^5\beta - \frac{2}{7}\sin^7\beta + \frac{1}{9}\sin^9\beta\right]$$

$$= -\frac{12}{5}f^2\sin^5\beta + \frac{3}{14}f^2\sin^7\beta - \frac{1}{12}f^2\sin^9\beta\Big|_{\beta_1}^{\beta_2} \qquad (11)$$

III:

$$\frac{3}{4}f^3\int\sin^2\beta\times 16\left(\sin^4\beta - 2\sin^6\beta + \sin^8\beta\right)\cos\beta\ d\beta$$

$$= 12f^3\int\left(\sin^6\beta - 2\sin^8\beta + \sin^{10}\beta\right)\cos\beta\ d\beta$$

$$= 12f^3\left[\frac{1}{7}\sin^7\beta - \frac{2}{9}\sin^9\beta + \frac{1}{11}\sin^{11}\beta\right]$$

$$= \frac{12}{7}f^3\sin^7\beta - \frac{24}{9}\sin^9\beta + \frac{12}{11}f^3\sin^{11}\beta\Big|_{\beta_1}^{\beta_2} \qquad (12)$$

IV:

$$\frac{9}{64}f^4\int 16\left(\sin^4\beta - 2\sin^6\beta + \sin^8\beta\right)\cos\beta\ d\beta$$

$$= \frac{9}{4}f^4\int\sin^4\beta\cos\beta\ d\beta - \frac{9}{2}f^4\int\sin^6\beta\cos\beta\ d\beta + \frac{9}{64}f^4\int\sin^8\beta\cos\beta\ d\beta$$

$$= \frac{9}{4}f^4\frac{1}{5}\sin^5\beta - \frac{9}{2}f^4\frac{1}{7}\sin^7\beta + \frac{9}{64}f^4\frac{1}{9}\sin^9\beta$$

$$= \frac{9}{20}f^4\sin^5\beta - \frac{9}{14}f^4\sin^7\beta + \frac{1}{64}f^4\sin^9\beta\Big|_{\beta_1}^{\beta_2} \qquad (13)$$

Putting it all together, Eq. (9) becomes:

$$\text{Area} = a^2 \Delta\lambda \Big[I + II + III + IV \Big]_{\beta_1}^{\beta_2} \tag{14}$$

where β_1 and β_2 indicate the latitude limits to the region of integration.

Now time for the sanity checks:

- First, in the case of a sphere, $f = 0$, and all terms disappear except the lead term in I:

$$\text{Area}_{\text{sphere}} = a^2 \Delta\lambda \sin\beta \Big|_{\beta_1}^{\beta_2}$$

- In this case, and for $\beta_2 = \pi/2$ and $\beta_1 = -\pi/2$ and $\Delta\lambda = 2\pi$, we wind up with the area of a sphere, $4\pi a^2$.
- Also, in the limiting case of one of the poles, this expression reduces to $a^2\Delta\lambda\Delta\beta/2$, which is the expected area of the little triangle.
- Second, in the limit of $\Delta\lambda \to 0$ and/or $\Delta\beta \to 0$, Area $\to 0$.
- Third, all terms in the final expression are odd, indicating symmetry around the equator, as expected.

Table of Sample Areas, in [km^2]

latitude [deg]	spheroid	sphere
89.75	3.808761	3.834437
89.25	11.425984	11.502999
88.75	19.042372	19.170685
88.25	26.657368	26.836910
87.75	34.270416	34.501092
87.25	41.880961	42.162646
46.25	605.586083	607.693848
45.75	611.119569	613.210359
45.25	616.607171	618.680173
44.75	622.048454	624.102871
44.25	627.442986	629.478041
43.75	632.790338	634.805275
31.25	749.956218	751.286874
30.75	753.937761	755.236626
30.25	757.861900	759.128863
29.75	761.728314	762.963290
29.25	765.536684	766.739614
28.75	769.286698	770.457548

2.25	878.102361	878.111163
1.75	878.373461	878.378803
1.25	878.576806	878.579550
0.75	878.712379	878.713390
0.25	878.780168	878.780312

The following IDL procedure evaluates our expression for the case of Mars (a = 3397 km):

```
pro SPHEROID_AREA
radius=3397.d0 ; equatorial, in km
f=0.00647630 ; flattening
gs=0.5d0 ; gridsize, in degrees
halfgs=gs/2.d0
outfile='spheroid_area.dat'
openw,unit,outfile,/get_lun
printf,unit,'latitude Area(km^2,sphere) Area(km^2,spheroid)'
for lat=90-halfgs,-90+halfgs,-gs do begin
top=lat+halfgs & bot=lat-halfgs
area=!dtor*gs*radius^2*(dotrig(top,f,gs)-dotrig(bot,f,gs))
sphere_area=!dtor*gs*radius^2*(sin(!dtor*top)-sin(!dtor*bot))
printf,unit,lat,area,sphere_area,format='(1x,f6.2,2(3x,f12.6))'
endfor
close,unit
print,'Done. Data written to ',strn(outfile)
DONE:
stop
end
function dotrig,lat,f,gs
f2=f*f & f3=f2*f & f4=f3*f
sinlat=sin(lat*!dtor) & sinlat2=sinlat*sinlat
sinlat3=sinlat^3 & sinlat5=sinlat^5
sinlat7=sinlat^7 & sinlat9=sinlat^9
sinlat11=sinlat^11
term1=sinlat-(f/3.d0)*sinlat3+(f2/5.d0)*sinlat5
term2=-(12.d0/5.d0)*f2*sinlat5+(13.d0/14.d0)*f2*sinlat7-(f2/12.d0)*sinlat9
term3=(12.d0/7.d0)*f3*sinlat7-(8.d0/3.d0)*f3*sinlat9+(12.d0/11.d0)*f3*sinlat11
term4=(9.d0/20.d0)*f4*sinlat5-(9.d0/14.d0)*f4*sinlat7+(f4/64.d0)*sinlat9
print,term1,term2,term3,term4
return,(term1+term2+term3+term4)
end
```